# Competition among Sellers by Mechanism Design

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June 2005

**Abstract**

In this paper we study a multistage game of competition among sellers in the design of trade mechanisms. First, sellers simultaneously choose trade mechanisms from the class of anonymous and incentive-compatible mechanisms. Second, upon observing the sellers' "offers", each buyer decides which seller to visit, if any. Third, buyers learn their valuations and the number of bidders participating in the same mechanism. Finally, bidders report their valuations, the mechanisms are operated and the transactions take place. We provide conditions for the existence of a pure strategy symmetric subgame perfect equilibrium mechanisms for any number of buyers and sellers. The equilibrium mechanisms are auctions with a trivial (zero cost) reservation price and an entrance fee. An equilibrium is derived in which the entrance fee is independent on the number of participating bidders. We derive the equilibrium fee as a function of the buyers' valuations and the value of the outside option. The paper contributes to the literature on the topic by proving the existence and solving for the subgame perfect equilibrium of the general mechanism design problem in the finite buyer and seller case.

*Key Words:* competition, mechanism design, auctions

*JEL Classification:* D44, D82.

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# 1 Introduction

In this paper we will address the question of what kind of trade mechanisms would arise in a *strategic* equilibrium of an oligopoly market, in which several sellers compete for a common pool of customers by designing *trade mechanisms*. Our analysis will be conducted within the independent private value model, in which buyers' valuations are privately observed and drawn from identical distributions. We will analyze a market game with the following time structure. In the first stage of the game sellers, who possess a single unit of a homogeneous good, simultaneously choose trade mechanisms from the class of anonymous and incentive-compatible mechanisms (see Myerson (1981)). In the second stage, upon observing the choice of the sellers, each buyer decides which seller to visit, if any. Randomizing over the sellers' mechanisms is allowed. In the third stage buyers learn their valuations and the number of their competitors participating at the same mechanism and submit their bids. Finally, the mechanisms are operated and the transactions take place.

The present inquiry belongs to a growing literature on mechanism design by competing sellers, which has been initiated by McAfee (1993). This literature studies the relationship between trade mechanisms offered by sellers and bidders' distribution across sellers as well as the consequences of this relationship for the sellers' choice of trade mechanisms in equilibrium.

It has basically two complex problems to deal with. The first one is to determine the equilibrium distribution of bidders across sellers for every profile of offered mechanisms. The second one is to solve for the equilibrium in sellers' trade mechanisms in the first stage of the game.

McAfee (1993) deals with the first problem by suggesting a new equilibrium concept, which he terms competitive subform consistent equilibrium (CSCE). It requires that every seller ignore his influence on the expected profits offered to buyers by other sellers. This assumption is applicable to a market with an infinite number of buyers and sellers, in which each seller's decision has no effect on the distribution of buyers across the other mechanisms, but is not appropriate for finite economies. Peters and Severinov (1997) propose a new limit equilibrium concept, competitive matching equilibrium (CME), which justifies McAfee's conjecture for a large number of market participants on both sides of the market.

The second problem relates to the complexity of the sellers' strategy space and is basically circumvented by significantly restricting the class of possible mechanisms. Burguet and Sakovics (1999), Peters and Severinov (1997) and Hernando-Veciana (2005) restrict the sellers' strategy space to second price auctions in which the sellers choose their reservation price. This assumption is convenient, because a trade mechanism can be described by a single variable. Burguet and Sakovics (1999) show that in the two-seller case reserve prices are not driven down to production costs and the mixed strategy[1] symmetric subgame perfect equilibrium (SPE) is inefficient. Hernando-Veciana (2005) demonstrates, that for any finite set of feasible reserve prices, reserve prices in a (SPE) go down to production costs if the numbers of auctioneers and bidders is sufficiently large, but finite.

In the present paper we will opt for neither of the solutions. On the one hand we will be interested in the (SPE) of the above described auction game, capturing all the repercussions that a change in a seller's mechanism has on the payoffs of the bidders with other mechanisms. On the other hand we will not be restricting the strategy space of the sellers, thus dealing directly with the general mechanisms design problem. As in McAfee (1993) we will find the equilibrium trade mechanisms are auctions: the object should go to the highest value bidder. The equilibrium auctions have zero (cost) reservation price, but involve an entrance fee, which might depend on the number of bidders. We provide conditions for the existence of a unique symmetric equilibrium in which the entrance fee does not depend on the number of participants. The participation fee is derived as a function of the distribution of buyer's valuations and the option of not entering the market. The paper contributes to the existing literature on the topic by solving the general mechanism design problem for the finite buyer and seller case. The use of second price auctions with entrance fees or any other payoff equivalent auction is here a derived result and not an assumption.

Generally two models have been suggested in the literature. The first one assumes that buyers learn their valuation after visiting a seller and inspecting the good on offer. The second one assumes that the buyers know their valuation prior to deciding which seller to visit. McAfee (1993), Burguet and Sakovics (1999) and Hernando-Veciana (2005) consider the former variant. Wolinsky (1988) considers the first one but in his model the matching technology of buyers and sellers is random and exogenous. Peters and Severinov (1997) analyze both cases for their limiting equilibrium concept. Both variants can be considered as benchmark cases. The former one is reasonable, if buyers need some time to study the good to form their valuation. If however bidders search for some predefined attributes, the latter one will be more appropriate. It is however much more difficult to analyze, since the decision to visit a particular seller depends additionally on the valuation of the bidder. How bidders with different valuations distribute over the sellers' mechanism according to their valuation in equilibrium is a difficult problem to solve. It still remains an open issue

[1]In their model the existence of pure strategy equilibria is not guaranteed.

in its finite version, in which sellers are free to choose any direct mechanism.

We show that the first problem is tractable even for a finite number of buyers and sellers and a very general sellers' strategy space. Peters and Severinov (1997, pp. 147-153) consider this case, but restrict the strategy space to second price auctions with a reserve price. Surprisingly, this restriction of the strategy space leads to problems with the existence of a (SPE). Burguet and Sakovics (1999, p. 240) provide an example for the nonexistence of a (SPE) in pure strategies in a market of two sellers and two buyers, whose valuation are drawn from a uniform distribution with support $[0, 1]$.

The paper is organized as follows. In the next section we will present the model: the general framework, the strategy space of buyers and sellers, their payoffs as well as the concept of a (SPE). Section 3 contains the derivation of the basic results and section 3.4 contains a numerical example for a (SPE) in a market of two sellers and two buyers. Section 4 concludes with a discussion of the results and their implications for the organization of some existing markets.

# 2 The model

## 2.1 Preliminaries

We consider an imperfectly competitive market with a number of $J \geq 2$ sellers (females) and $I \geq 2$ buyers (males). All agents are risk neutral. Each seller possesses a unit of a commodity, which she wishes to sell to a buyer. The use value equals across sellers and without loss of generality is normalized to zero. The sellers compete in the market by simultaneously choosing trade mechanisms. After observing the sellers' "offers" the buyers either choose a seller, whose trade mechanism to participate in or stay out of the market exercising an outside option. Buyers are allowed to participate only in one trading mechanism, but randomizing over the sellers' trade mechanism and the outside option is possible. Exercising the outside option[2] is associated with a sure payoff of $\beta \geq 0$ for a buyer. Once a buyer selects a seller, he learns his valuation, which is a draw from a random variable. After learning his valuation and the number of the bidders participating in his mechanism, bidders participate in the mechanism by reporting their type. Finally, the resulting allocation is implemented.

[2]The outside option can be broadly understood. It might for example be associated with the transportation costs to arrive at the marketplace or the costs which a bidder spends to learn his own valuation.

## 2.2 Notation

Buyers will be indexed by $i$ and sellers by $j$. The valuation of buyer $i$, $x_i$ is private information and a random draw from the interval $[0,1]$ according to the continuously differentiable distribution function $F$. If a buyer participates in the mechanism of a certain seller, he will be asked to report his private valuation to the mechanism. Since the number of bidders visiting certain seller is not known ex-ante, the mechanism should prescribe an allocation and a payment rule for any number (and identity) of bidders and any realization of their valuations. Let us denote for that purpose the set of the subsets (the power set) of all bidders by $\mathcal{I}$ and the power set of all rivals of bidder $i$ by $\mathcal{I}^{-i}$. Let $s \in \mathcal{I}$ denotes a group of bidders and $x^s$ the ordered vector[3] of the valuations of bidders from the group $s$. Further let $X^s$ denote the set of all possible ordered vectors of their valuations. Let

$$X \equiv \bigcup_{s \in \mathcal{I}} X^s$$

denote the set of all ordered vectors of the valuations of all subsets of bidders and $x$ an element of this set[4]. Similarly by $X_{-i}$ one denotes the set of the ordered vectors of the valuations of all subsets of bidders except bidder $i$.

## 2.3 Sellers' strategy space

Before defining the sellers' strategy set, let us first denote the set

$$\mathfrak{A} \equiv \Big\{ \{(p_i, z_i)\}_{i \in I} \mid (p_i, z_i) : X \to [0,1] \times \mathbb{R} \Big\},$$

which contains all trade mechanisms satisfying the conditions (NP),(F) and (A) given below. Generally, a trade mechanism is defined by the functions $p_i(\cdot)$ for every bidder $i$, which determine the probability, with which every bidder $i$ receives the item and by the functions $z_i(\cdot)$, determining the payment of every bidder $i$ to the mechanism.

- *Non-participation condition:*

$$p_i(x^s) = 0, \quad z_i(x^s) = 0, \forall i \notin s. \tag{NP}$$

  The condition requires that bidders who do not participate in a certain mechanism don't win the object and don't pay.

[3]By ordered vector $x^s$ we refer to the vector of the valuations of the bidders from a subset $s$, in which the components are ordered in an ascending order according to the bidder's number.

[4]Note that the valuation of each bidder $i$, $x_i$, might or might not appear in the vector $x$ depending on whether this bidder $i$ participates in the mechanism or not.

- *Feasibility:*

$$\sum_{i=1}^{I} p_i(x) \leq 1, \quad \forall x \in X. \tag{F}$$

The feasibility conditions requires that for any realization of the private information of the participating bidders the mechanism rules do not allow more units to be sold than physically available. Here we allow also for mechanisms for which for some realizations of $x$ the inequality can be satisfied. This might for example be the case if the mechanism is a second-price auction with a positive reservation price. I such a case if the valuation of the participating bidders lie below the seller's reserve price, she will retain the item.

- (A) *Anonymity*:

Let

$$p(\cdot) \equiv \Big(p_1(\cdot), p_2(\cdot), \dots, p_I(\cdot)\Big) \quad \text{and} \quad z(\cdot) \equiv \Big(z_1(\cdot), z_2(\cdot), \dots, z_I(\cdot)\Big)$$

denote the vectors of probability and allocation functions (respectively). The anonymity condition requires that the functions $p$ and $z$ are *permutation invariant.* This means that permuting the valuations of any ordered vector $x \in X$ permutes the vectors $p(x)$ and $z(x)$ in the same fashion. Let $(x_k, x_l, x^s)$ denote the ordered vector of the valuation of the bidders from the group $s$ and the bidders $l, k \notin s$. Then the permutation invariance implies:

$$\begin{aligned}
p_k(x_k, x_l, x^s) &= p_l(x_l, x_k, x^s), \\
z_k(x_k, x_l, x^s) &= z_l(x_l, x_k, x^s), \\
p_i(x_k, x_l, x^s) &= p_i(x_l, x_k, x^s), \\
z_i(x_k, x_l, x^s) &= z_i(x_l, x_k, x^s),
\end{aligned}$$

$\forall l, k, \forall i \in s, \forall s$. The anonymity or equal treatment guarantees that sellers do not discriminate among buyers on characteristics different than their reports to the mechanism or in other words the chances of winning and the payment are not dependent on the buyers' identities but solely on their reports to the mechanism. McAfee (1993) and Peters (1994) also consider anonymous mechanisms and provide equivalent definitions.

The set $\mathfrak{A}$, satisfying the above conditions, is larger than the sellers' strategy set. We will further narrow down the set of possible mechanisms among which the sellers can choose by imposing additional conditions on the probability and payment functions.

To define the sellers' strategy space, which we will denote by $\widehat{\mathfrak{A}}$, we impose the two additional requirements.

- *Incentive compatibility:*
  Let us assume that bidder $i$ chooses to participate in mechanism $j$. He learns his valuation $x_i$ (by inspecting the object for sale for example) and the fact that he will compete for the object with the buyers from the set $s \in \mathcal{I}^{-i}$. We denote the expected probability of winning and the expected payment of bidder $i$, who reports the valuation $\tilde{x}_i$, provided that the other participants report truthfully by

$$P^s(\tilde{x}_i) := \int p_i(\tilde{x}_i, x^s) dF(x^s),$$
$$Z^s(\tilde{x}_i) := \int z_i(\tilde{x}_i, x^s) dF(x^s).$$

  The incentive compatibility requires, that bidder $i$ finds it profitable to report truthfully if all other bidders do so, i.e. for every $s \in \mathcal{I}^{-i}$ and every $\tilde{x}_i \in [0,1]$ the following inequality holds:

$$\begin{aligned} E^s(\tilde{x}_i \mid x_i) &\equiv x_i \cdot P^s(\tilde{x}_i) - Z^s(\tilde{x}_i) \\ &\leq x_i \cdot P^s(x_i) - Z^s(x_i) \equiv E^s(x_i \mid x_i) =: E^s(x_i). \end{aligned} \tag{IC}$$

  $E^s(\tilde{x}_i \mid x_i)$ is the expected payment of a bidder, who has a valuation of $x_i$ and reports the valuation $\tilde{x}_i$ to the mechanism.

  There is indeed no loss of generality to restrict the sellers to use incentive compatible mechanisms. In the present setting buyers submit bids after they learn their valuations and the number of their fellow bidders (but not their valuations), so the sellers' mechanisms described here are standard Bayesian games for which the *revelation principle* applies (see e.g. Myerson (1997, p. 260)).

- (R) *Regularity:*
  Let us assume that bidder $i$ participates in a certain mechanism $j$ with a probability of one and let all other bidders visit this mechanism with a probability of $m$. The regularity condition requires that the expected payoff of a bidder from participating in the mechanism $j$ is (weakly) decreasing in the probability $m$. A formal definition will be given later on after we define the bidders' strategy space and their expected payoff. Roughly speaking, the condition requires that a bidder's expected payoff decreases with increased competition for this mechanism.

## 2.4 Bidders' strategy space

Conditional on observing the mechanisms offered by the sellers, the bidders decide which seller to visit. Although bidders are not allowed to visit more than one seller, randomizing over the sellers is allowed. Thus, the bidders play a *behavior strategy* as they decide on every node defined by a profile of trade mechanisms of the sellers, which seller to visit. Eichberger (1993, pp. 22-24) offers a definition and a discussion on the behavior strategy concept. The strategy of bidder $i$ is a mapping from the set of possible vectors of trade mechanism into probabilities, with which that bidder plans to visit each seller. We will denote a strategy of bidder $i$ by

$$m^i = \Big(m^i_o(\cdot), m^i_1(\cdot), m^i_2(\cdot) \ldots, m^i_J(\cdot)\Big),$$

where

$$m^i_j : \widehat{\mathfrak{A}}^J \to [0,1]; \quad m^i_o : \widehat{\mathfrak{A}}^J \to [0,1] \quad \text{and} \quad m^i_o(\cdot) + \sum_{j=1}^{J} m^i_j(\cdot) = 1.$$

It will be useful to represent a strategy profile of the bidders by the $I \times (J+1)$ matrix

$$m(\cdot) := \begin{pmatrix} m^1_o(\cdot) & m^1_1(\cdot) & m^1_2(\cdot) & \ldots & m^1_J(\cdot) \\ m^2_o(\cdot) & m^2_1(\cdot) & m^2_2(\cdot) & \ldots & m^2_J(\cdot) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^I_o(\cdot) & m^I_1(\cdot) & m^I_2(\cdot) & \ldots & m^I_J(\cdot) \end{pmatrix}.$$

A strategy profile of all bidders except bidder $i$ will be denoted by $m^{-i}(\cdot)$. We will say, that the bidders use a symmetric behavioral strategy, if the functions in every column of the matrix are identical. A symmetric strategy profile will be denoted by

$$\big(m_o(\cdot), m_1(\cdot), m_2(\cdot), \ldots, m_J(\cdot)\big).$$

## 2.5 Payoffs

Let us denote by $m^{-i}_j(p,z)$ the vector of probabilities with which all bidders except $i$ visit mechanism $j$ (this is the $j-$th column in the above matrix, except the probability of bidder $i$). If bidder $i$ visits mechanism $j$ with a probability of one, then his payoff is given by

$$R^i_j\Big((p^j, z^j); m^{-i}_j(p,z)\Big) = \sum_{s \in \mathcal{I}^{-i}} \prod_{l \in s} m^l_j(p,z) \cdot \int_0^1 E^s_j(x_i) dF(x_i).$$

In the payoff of bidder $i$ one sums the products of the probabilities with which bidder $i$ encounters any group of rivals and his expected payoff in case that this group of rivals is encountered. The payoff of seller $j$ is

$$\Pi_j\Big((p,z);m(p,z)\Big)=\sum_{i=1}^{I}m_j^i(p,z)\cdot\Big(\sum_{s\in\mathcal{I}^{-i}}\prod_{l\in s}m_j^l(p,z)\cdot\int_0^1 Z_j^s(x_i)dF(x_i)\Big),$$

which is the sum of the expected payments of the bidders to the mechanism $j$. Since we consider anonymous mechanisms, the functions $E_j^s(x_i)$, $Z_j^s(x_i)$ and $P_j^s(x_i)$ depend only on the number of rivals of bidder $i$ (in the set $s$) and not on their identity. Therefore for simplicity from now on we will use the notation $E_j^{(n)}(x_i)$, $Z_j^{(n)}(x_i)$ and $P_j^{(n)}(x_i)$ when describing the payoff, the payment and the probability with which bidder $i$ is served when he faces $(n-1)$ rivals. If all rivals of bidder $i$ visit mechanism $j$ with a probability of $m$, then his payoff from participating with a probability of one is

$$R_j^i\Big((p^j,z^j);m\Big)=\sum_{n=1}^{I}\binom{I-1}{n-1}m^{n-1}(1-m)^{I-n}\cdot\int_0^1 E_j^{(n)}(x_i)dF(x_i).$$

If all bidders visit mechanism $j$ with probability $m$, then the expected profit of seller $j$ is

$$\Pi_j\Big((p^j,z^j);m\Big)=\sum_{n=1}^{I}\binom{I}{n}m^{n}(1-m)^{I-n}\cdot\int_0^1 Z_j^{(n)}(x_i)dF(x_i).$$

Now, we can formally define the regularity condition introduced in subsection (2.3).

**Definition 1 (R).** *A mechanism $(p^j,z^j)$ is **regular** if the function $R_j^i\Big((p^j,z^j);m\Big)$ is (weakly) decreasing in $m$.*

The regularity condition is satisfied by the standard auction formats. We will show that all payoff equivalent mechanisms to a second price auction with an entrance fee, which does not depend on the number of participating bidders, are regular mechanisms[5] (see lemma 2). The regularity condition is not satisfied for example by mechanisms according to which the seller imposes high participation fees if a low number of bidders participate and a low participation fee (or even a bonus) if many bidders visit the mechanism. In such a situation an increased competition can lead to higher expected payoffs for the participants.

If bidder $i$ employs the behavioral strategy $m^i(p,z)$, his payoff is:

$$\mathcal{R}_i\Big((p,z);m(p,z)\Big)=m_o^i(p,z)\cdot\beta+\sum_{j=1}^{J}m_j^i(p,z)\cdot R_j^i\Big((p^j,z^j),m_{-i}(p,z)\Big).$$

[5]One can easily show that the second price auction with a non-trivial reserve price is a regular mechanism as well (the proof of this claim emulates the proof of lemma 2, which is given in Appendix A).

As already indicated, after arriving at the mechanism, buyers learn their valuation (for example by inspecting the item for sale) and the number of their fellow bidders participating at that mechanism. As we require that the mechanisms are incentive-compatible, bidders report their valuations truthfully, the mechanisms are operated and the transactions take place.

### 2.6 Equilibrium

In this model we will be interested in the symmetric subgame perfect equilibria of the model, which are defined as follows.

**Definition 2.** *The sellers' strategy profile* $(p^*, z^*)$ *and the symmetric selection behavioral strategy functions of the bidders represented by the matrix* $m^*(\cdot)$ *constitute a* ***symmetric subgame perfect equilibrium*** *(short: equilibrium), if they satisfy the following conditions:*

1. *(Optimal selection by buyers):*

$$\mathcal{R}_i\big((p,z); m^{*i}(p,z), m^{*-i}(p,z)\big) \geq \mathcal{R}_i\big((p,z); m^i, m^{*-i}(p,z)\big),$$
$$\forall (p,z) \in \widehat{\mathfrak{A}}^J, \forall i, \forall m^i \in [0,1].$$

2. *(Nash equilibrium play in the reduced form of the game):*

$$\Pi_j\Big((p^{*j}, z^{*j}), (p^{*-j}, z^{*-j}); m^*(\cdot)\Big) \geq \Pi_j\Big((p^j, z^j), (p^{*-j}, z^{*-j}); m^*(\cdot)\Big), \forall (p^j, z^j) \in \widehat{\mathfrak{A}}. \tag{NE}$$

3. *(Symmetry): All sellers use the same trade mechanism.*

The first equilibrium condition requires that in each subgame defined by the sellers' choice of mechanisms the bidders randomize symmetrically over the mechanisms, i.e. they play symmetric behavior strategies, which constitute a Nash equilibrium in the second stage of the game. The second condition requires that the sellers choose mechanisms, which build a Nash equilibrium in the first stage of the game.

## 3 Analysis

### 3.1 Organization of the analysis and results

In this work we will show that in equilibrium sellers hold auctions (Theorem 1). Holding auctions in this setting amounts to using a trade mechanism, which assigns the unit to

| | $\mathfrak{A}$ | $\widehat{\mathfrak{A}}$ | $\Omega$ | $\widehat{\Omega}$ |
|---|---|---|---|---|
| Non-Participation | + | + | + | + |
| Feasibility | + | + | + | + |
| Anonymity | + | + | + | + |
| Incentive Compatibility | | + | + | + |
| Regularity | | + | + | + |
| Efficiency | | | + | + |
| Constant entrance fee | | | | + |

**Table 1:** *Mechanism sets and conditions.*

the participant with the highest valuation (see McAfee (1993, p. 1292) and the exposition of the next subsection). To summarize the results and explain the arguments behind the proofs Table 1 will be helpful. The rows in this table represent conditions imposed on trade mechanisms. The first five conditions are already defined. The "Efficiency" condition requires that the object should always be granted to the participant with the highest valuation. The "Constant entrance fee" condition requires that the seller uses an auction with an entrance fee, which does not depend on the number of bidders participating at the mechanism. The + sign denotes which conditions are satisfied by the mechanisms from the sets $\mathfrak{A}$, $\widehat{\mathfrak{A}}$, $\Omega$ and $\widehat{\Omega}$. The sets $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ are already defined. $\widehat{\mathfrak{A}}$ is the sellers' strategy set. Lemma 1 derives the profit maximizing mechanisms (for a seller) among all mechanisms from the set $\mathfrak{A}$ which give every participating bidder a constant expected payment (of $R^*$), provided that all bidders visit this mechanism with a certain probability (of $m^*$). It states that the profit-maximizing mechanism should be efficient, i.e. the object should go to the participant with the highest valuation. As a consequence of the lemma one narrows down substantially the set of mechanism, which can constitute an equilibrium in the game with strategy set $\widehat{\mathfrak{A}}$. After imposing additionally the (IC) and (R) conditions only the mechanisms from the set $\Omega$ remain as possible equilibria. This set consists only of auctions with a zero-reserve price and an entrance fee, which might depend on the number of the bidders (this argument rests on a standard results from the theory). In the exposition later on we will discuss this argument in more detail. We further are restricting attention only to the set $\widehat{\Omega}$, which consists of auctions with an entrance fee independent on the number of bidders visiting the mechanism. In Theorem 2 we show however that if a strategy profile of the sellers is an equilibrium in the game with strategy space $\widehat{\Omega}$, it is also an equilibrium in the game with a strategy space $\Omega$. It follows that this equilibrium strategy profile constitutes also an equilibrium with the game with strategy space $\widehat{\mathfrak{A}}$. Theorem 3 provides conditions for the existence of equilibrium (within the set $\widehat{\Omega}$) and characterizes the equilibrium trade mechanism.

## 3.2 Theorems and proofs

**Theorem 1.** *(i) The sellers' equilibrium mechanisms (provided that an equilibrium exists) assigns the item (almost surely) to the highest-valuation bidder, if this valuation is higher than the seller's use value.*

*(ii) The equilibrium mechanisms are payoff equivalent to a second price auction with a reserve price equal to the seller's valuation and an entrance fee, which might depend on the number of participating bidders.*

This theorem establishes some equilibrium properties without resolving the question of existence of an equilibrium. We shall deal later on with this problem by providing conditions which guarantee the existence and uniqueness of a symmetric equilibrium in this market game for an arbitrary number of sellers and buyers. The present theorem is useful, as it restricts the type of mechanism profiles, which can constitute an equilibrium. This initial result will further be employed for the characterization of equilibrium and for the existence and uniqueness proof itself.

*Proof.* For part *(i)* we will proceed by contradiction. Take an equilibrium profile $(p^*, z^*)$ and let in this equilibrium buyers visit a certain mechanism $j$ with probability $m^*$. Let the expected profit of a buyer participating in the mechanism of seller $j$ be denoted by $E^*$. We will show that if the equilibrium mechanism $(p^{j*}, z^{j*})$ does not satisfy the condition of part *(i)* of the theorem, one can construct a deviation mechanism $(p^{jD}, z^{jD}) \in \widehat{\mathfrak{A}}$ which assigns the object to the highest valuation bidder and does not change the equilibrium probability with which buyers visit that seller. We show that this mechanism is more profitable for the seller, reaching a contradiction to the equilibrium requirement (NE). We start with the following lemma.

**Lemma 1.** *The profit maximizing mechanisms for an arbitrary seller $j$, among all mechanisms from the set $\mathfrak{A}$ which give every participating bidder an expected payment of $R^*$, provided that all bidders visit this mechanism with probability $m^*$, assign the object with probability one to the participant with the highest valuation, if this valuation exceeds the seller's use value.*

A formal proof of the lemma is provided in the Appendix A. The statement is closely related to an argument provided by McAffee and McMillan (1987*b*), which concerns a setting with one seller and an outside option. Their argument is useful to understand the idea of the current proof and therefore will be shortly sketched here. For any number of participating bidders the seller's expected revenue is the winning bidder's expected valuation minus the expected profit of the participating bidders. Thus, for any given number of participating bidders the seller should award the good so as to maximize the expected valuation of the winner. This can only be done by awarding the good to the highest valuation bidder whenever this valuation exceeds the seller's reserve value[6].

Observe that the lemma does not require that the mechanisms satisfy the incentive compatibility constraint (IC) or the regularity condition (R). If we found a deviation mechanism $(p^{jD}, z^{jD})$ which belongs to the set $\widehat{\mathfrak{A}}$ (i.e. satisfies additionally the conditions (IC) and (R)) and does not change the probability distribution of buyers across sellers, then from the lemma would follow that this deviation is profitable.

Observe that a bidder participating in the mechanism of seller $j$ might face any number of 0 to $I-1$ bidders. Let $n \in \{1, 2, \ldots, I\}$ denote the total number of bidders participating in the mechanism of that seller $j$. The winning probability of bidder $i$ with valuation $x_i$ who reports valuation $\tilde{x_i}$, if the other $(n-1)$ bidders report truthfully is $P_i^n(\tilde{x_i}) \equiv [F(\tilde{x_i})]^{n-1}$. By the Envelope theorem one obtains for the derivative of the payoff of bidder $i$ at an incentive-compatible mechanism[7] which awards the good to the highest valuation bidder:

$$\frac{d}{dx_i}(E^{(n)}(\tilde{x_i} \mid x_i)) = \frac{\partial}{\partial x_i}(E^n(\tilde{x_i} \mid x_i))\bigg|_{\tilde{x_i}=x_i} = [F(x_i)]^{n-1}.$$

The expected profit is

$$E^{(n)}(x_i \mid x_i) = C_n + \int_0^{x_i} [F(x_i)]^{n-1} dx_i,$$

where $C_n$ is the expected profit of a bidder with the lowest valuation 0. $C_n$ is thus the entrance fee or bonus, which each bidder has to pay or receives, when participating in the mechanism with $(n-1)$ other bidders. From the theory of optimal auctions[8] it is know that the (ex ante) expected payment of a bidder participating with $(n-1)$ other bidders in an incentive-compatible mechanism with 0 entrance fee, which assigns the object to the highest value bidder is

$$B_n = \int_0^1 \int_0^x [F(z)]^{n-1} dz dF(x).$$

The expected profit of a seller who auctions an item to $n$ bidders is

$$S_n = n \cdot \int_0^1 [x \cdot f(x) + F(x) - 1] \cdot [F(x)]^{n-1} dx.$$

[6] In our setting the seller's reserve value is 0 and the bidders' valuations are distributed on the interval $[0, 1]$, so they are almost surely higher than the seller's reserve value.

[7] Here we follow the exposition in McAffee and McMillan (1987*b*).

[8] See for example Riley and Samuelson (1981) or McAffee and McMillan (1987*a*).

All these mechanism are payoff equivalent to a second price auction with a zero reservation price.

A mechanism from the set $\Omega$ can now be identified only by the participation fee for any number of participants and will be denoted by $(C_1, C_2, \ldots, C_I)$. For constructing the deviation mechanism consider a mechanism $(\underbrace{C, C, \ldots, C}_{I})$ requiring the same participation fee independent on the number of the bidders[9]. Consider the following lemma.

**Lemma 2.** *All incentive compatible mechanisms involving an entrance fee, which is independent on the number of participants, are* ***regular*** *mechanisms.*

The proof is somewhat technical and not of interest in itself. It is moved to Appendix A. The lemma guarantees that this deviation mechanism indeed belongs to the set $\widehat{\mathfrak{A}}$. The expected profit of a bidder from participating in the deviation mechanism, if every other bidder participates with probability $m^*$ is

$$E^* = \sum_{n=0}^{I-1} B_{n+1} \cdot \binom{I-1}{n} (m^*)^n (1-m^*)^{I-1-n} - C.$$

Choosing a participation fee of

$$\sum_{n=0}^{I-1} B_{n+1} \cdot \binom{I-1}{n} (m^*)^n (1-m^*)^{I-1-n} - E^*$$

would present the desired deviation mechanism. It remains to be verified, that this deviation will not reshuffle the probability distribution of bidders across sellers. This is guaranteed by the regularity condition imposed on the strategy set $\widehat{\mathfrak{A}}$. This condition precludes the cases in which a bidder participating in a certain mechanism obtain the same payoff in cases in which the other bidders visit this mechanism with a different probability. Indeed, if each bidder visits mechanism $j$ with a probability higher than $m^*$, then the expected profit of each bidder will fall below $E^*$, whereas the expected profit with other mechanism will increase above $E^*$. On the other hand, if each bidder visits mechanism $j$ with a probability lower than $m^*$, then the bidders' expected profit with $j$ will rise and the expected profit with other mechanisms will fall. In sum, the bidders' selection stage of the game will not be in equilibrium for any probability of visiting $j$, which is different than $m^*$. □

Recall that we denoted the set of incentive compatible and regular mechanisms, in which the highest valuation bidder wins and the entrance fee is independent on the number of participating bidders by $\widehat{\Omega}$. One can state the following theorem.

[9] There are many ways to construct a deviation mechanism. This is one of the variants.

**Theorem 2.** *If the sellers' strategy profile*

$$\Big(\underbrace{\underbrace{(C^*, C^*, \ldots, C^*)}_{I}, \underbrace{(C^*, C^*, \ldots, C^*)}_{I}, \ldots, \underbrace{(C^*, C^*, \ldots, C^*)}_{I}}_{J}\Big)$$

*is an equilibrium profile of the game with strategy space* $\widehat{\Omega}$*, then it is also an equilibrium profile in the game with strategy space* $\Omega$*.*

The theorem is useful for the proof of the existence of equilibrium in the original game (with a strategy set $\widehat{\mathfrak{A}}$). The next theorem will assert that the game with strategy space $\widehat{\Omega}$ has an equilibrium. From the present theorem and theorem 1 follows then that the original game has the same equilibrium profile.

*Proof.* Take an equilibrium strategy $\underbrace{(C^*, C^*, \ldots, C^*)}_{I}$ and assume by a way of contradiction that there exists a profitable deviation of an arbitrary seller $j$, which we denote by $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I) \in \widehat{\mathfrak{A}}$. Let us assume that this strategy induce an equilibrium participation probability of $\tilde{m}$ and as the deviation is profitable we have

$$\Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) > \Pi_j\Big(\underbrace{(C^*, C^*, \ldots, C^*)}_{I}; m^*\Big).$$

The expected profit of bidder $i$ is

$$R_j^i\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) = \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot B_n - \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot \tilde{C}_n$$

and of the seller

$$\Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) = \sum_{n=1}^{I} \binom{I}{n} \tilde{m}^{n}(1-\tilde{m})^{I-n} \cdot S_n + \sum_{n=1}^{I} \binom{I}{n} \tilde{m}^{n}(1-\tilde{m})^{I-n} \cdot n \cdot \tilde{C}_n.$$

Since we consider only *regular* mechanisms the strategy $\underbrace{(\tilde{C}, \tilde{C}, \ldots, \tilde{C})}_{I} \in \widehat{\mathfrak{A}}$, where

$$\tilde{C} = \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1}(1-\tilde{m})^{I-n} \cdot \tilde{C}_n$$

induces the same unique participation probability $\tilde{m}$ and leads to the same expected bidder's payoff:

$$R^i_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) = R^i_j\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big).$$

The payoff of the seller is

$$\begin{aligned}\Pi_j\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big) &= \sum_{n=1}^{I} \binom{I}{n} \tilde{m}^n (1-\tilde{m})^{I-n} \cdot S_n \\ &\quad + I \cdot \tilde{m} \cdot \tilde{C}.\end{aligned}$$

One can readily observe now that

$$\begin{aligned}\sum_{n=1}^{I} \binom{I}{n} \tilde{m}^n (1-\tilde{m})^{I-n} \cdot n \cdot \tilde{C}_n &= I \cdot \tilde{m} \cdot \sum_{n=1}^{I} \binom{I-1}{n-1} \tilde{m}^{n-1} (1-\tilde{m})^{I-n} \cdot \tilde{C}_n \\ &= I \cdot \tilde{m} \cdot \tilde{C} \qquad\qquad \Leftrightarrow \\ \Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) &= \Pi_j\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big).\end{aligned}$$

Indeed, $(\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I})$ is so constructed that the expected payment of each bidder to seller $j$ equals the expected payment under the deviation strategy $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I)$ if both mechanisms are visited with the same probability of $\tilde{m}$. Therefore the expected fees that the seller obtains from every bidder are equal in both mechanisms. The resulting (in)equalities

$$\Pi_j\Big((\underbrace{\tilde{C}, \tilde{C}, \ldots, \tilde{C}}_{I}); \tilde{m}\Big) = \Pi_j\Big((\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_I); \tilde{m}\Big) > \Pi_j\Big((\underbrace{C^*, C^*, \ldots, C^*}_{I}); m^*\Big)$$

establish the desired contradiction to the equilibrium assumption.

To summarize, we assumed by contradiction that a certain strategy profile is an equilibrium of the game with the strategy space $\widehat{\Omega}$ and not an equilibrium of the game with the strategy space $\Omega$. As a consequence for one bidder a profitable deviation strategy from the set $\Omega$ exists. Then however we demonstrate that a profitable deviation strategy from the set $\widehat{\Omega}$ also exists, which poses a contradiction to the equilibrium assumption. □

The next theorem will characterize the equilibria for the game with strategy set $\widehat{\Omega}$. For that purpose we will introduce some addition notation. Let us denote the expected payoff of a bidder participating in a second price auction with a zero entrance fee if all other bidders visit this mechanism with a probability of $m$ by

$$R_{(m)} := \sum_{n=1}^{I} \binom{I-1}{n-1} (m)^{n-1} \cdot (1-m)^{I-n} \cdot B_n.$$

Let all sellers except seller $j$ play the strategy $(\underbrace{C, C, \ldots, C}_{I})$, and let seller $j$ employ the strategy $(\underbrace{C^j, C^j, \ldots, C^j}_{I})$. Let $\overline{m}(C^j, C)$ be defined as the solution of the equation

$$R_{(m)} - C^j = R_{(\frac{1-m}{J-1})} - C$$

and $\underline{m}(C^j, \beta)$ be the solution of the equation

$$R_{(m)} - C^j = \beta.$$

The function $\overline{m}(C^j, C)$ determines the probability with which bidders visits seller $j$, provided that they use the outside option with a probability of 0. The function $\underline{m}(C^j, \beta)$ determines the probability with which bidders will visit seller $j$, provided that they randomize between the outside option and the mechanism of seller $j$. Let $\Pi_j(C^j; m)$ denote the payoff of seller $j$ if she holds an auction with a participation fee of $C^j$ (independent on the number of participants) and all other bidders visit this mechanism with a probability of $m$. Define the functions

$$\overline{\varphi}(C) := \frac{\partial \Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)}{\partial C^j}\Bigg|_{C^j = C}$$

and

$$\underline{\varphi}(C^j, \beta) := \frac{\partial \Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)}{\partial C^j}.$$

**Theorem 3 (equilibrium).** *The game with strategy space $\widehat{\Omega}$ has a unique symmetric subgame perfect equilibrium in pure strategies if the functions $\Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)$ and $\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$ are concave with respect to $C^j$. The equilibrium fee is*

$$C^*(\beta) = \begin{cases} \overline{C} & \text{for} \quad \beta \leq R_{(1/J)} - \overline{C}, \\ \max\{R_{(1/J)} - \beta, \underline{C}(\beta)\} & \text{for} \quad \beta > R_{(1/J)} - \overline{C}, \end{cases}$$

*where $\overline{C}$ is the unique solution of the equation $\overline{\varphi}(C) = 0$ and $\underline{C}(\beta)$ is the unique solution of the equation $\underline{\varphi}(C^j, \beta) = 0$.*

See Appendix B for a proof and figure 1 for a graphical illustration. Next we will investigate which markets satisfy the premises of the above theorem.

## 3.3 Concavity of the payoff functions

Theorem 3 provides a condition for the existence and uniqueness of equilibrium, which requires that the payoff functions $\Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)$ and $\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$ are concave

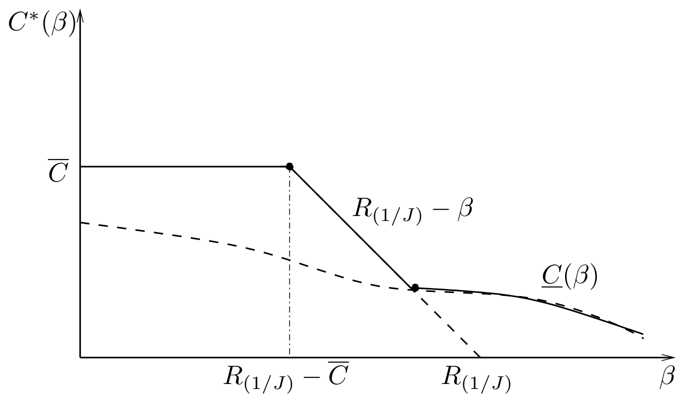


**Figure 1:** *The unique equilibrium entrance as a function of the outside option (the solid line).*

in $C_j$. Although we could not find an example in which this property is not satisfied, we also could not show that this property is satisfied for all probability distributions $F$ and any number of buyers and sellers. In this section we will show that in small markets (in the cases of two sellers and two or three buyers) the functions are convex for all $F$ and there always exists a unique equilibrium. A unique equilibrium exists also for any number of at least up to 100 buyers and sellers if $F$ is uniformly distributed. If $\beta$ is sufficiently high and $F$ is uniformly distributed an equilibrium exist for any number of buyers and sellers. The next two theorems establish these results.

**Theorem 4.** *In the cases $J = 2$ and $I \in \{2, 3\}$ the functions $\Pi_j\Big(C^j; \overline{m}(C, C^j)\Big)$ and $\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$ are concave in $C^j$ for any distribution $F$.*

**Theorem 5.** *If $F$ is uniformly distributed over the unit interval the function*

$$\Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)$$

*is concave for any number of $I \geq 2$ buyers and $J \geq 2$ sellers.*

The proofs are in Appendix B.

## 3.4 Numerical example: two buyers and two sellers

**Claim 1.** *If two sellers compete for two buyers (i.e. I=J=2) the equilibrium entrance fee is*

$$C^*(\beta) = \begin{cases} S_2/2 & \text{for } 0 \leq \beta < B_2, \\ B_2 + S_2/2 - \beta & \text{for } B_2 \leq \beta \leq B_2 + S_2/2, \\ 0 & \text{for } \beta > B_2 + S_2/2. \end{cases}$$

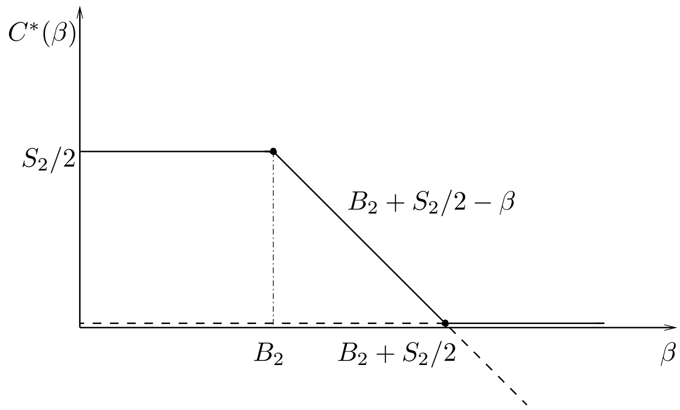


**Figure 2:** *Entrance fee in a subgame perfect equilibrium in the case $I = J = 2$ (the solid line).*

See Appendix B for a proof and Figure 2 for a graphical illustration of the equilibrium entrance fee. The equilibrium probability with which bidders visit an arbitrary seller $j \in \{1, 2, \ldots, J\}$ is given as follows[10]:

$$m_j^*(\beta) = \begin{cases} 1 & \text{for } 0 \leq \beta < B_2 + S_2/2, \\ (B_1 - \beta)/(B_1 - B_2) & \text{for } B_2 + S_2/2 \leq \beta \leq B_1, \\ 0 & \text{for } \beta > B_1. \end{cases}$$

Observe that for $\beta \in [B_2 + S_2/2, B_1]$ in equilibrium sellers lower the participation fee to allow all bidders to enter the market with a probability of one. This holds true until the entrance fee falls down to 0. As $\beta$ further increases sellers hold zero-reserve auctions with no entrance fee (or bonus). The entry probability in the market decreases linearly as $\beta$ further increases. For $\beta > B_1$ bidders do not enter the market any more.

# 4 Concluding remarks

The classical auction model studies the mechanism design problem of a monopoly seller in an environment of incomplete information regarding the valuations of the bidders. The present paper departs from this framework by considering a model of two or more sellers, which compete for the same pool of customers by designing trade mechanisms.

The primary message of the paper is that in a market of finitely many of buyers and sellers the equilibrium trade mechanisms will be auctions with a trivial reserve price. The model can be considered as a complement to McAfee's (1993) pioneering work, in which similar result is obtained in a model describing an infinite economy.

[10]See Appendix B.

Such a situation is evident in many markets. In housing markets close substitutes are sold via auctions; auction houses compete by selling similar products; on internet sites such as Ebay or Amazon sellers offer identical commodities like cameras, computers and other standardized products using a variety of sale methods: posted price, English auctions, Dutch auctions, auctions with a buy-it-now option, auctions with secret reserve prices, auctions with different closing rules, etc. Generally, the trade mechanism appears to be an important instrument in the competition for customers along the characteristics of the offered product.

Auctions with a trivial reserve price, called *absolute auctions*, are used as a sale method for instance in markets for restaurant equipment and real estate. Manning (2000), a real estate and restaurant equipment auctioneer, asserts that in his experience the public response to a property sold via an absolute auction is much more enthusiastic than similar property offered at an auction with a reserve price. Another (historical) example underscoring the benefits of an absolute auction is the rapid growth of trade through the Port of New York relative to the trade through other East Coast ports of the United States following the War of 1812. Engelbrecht-Wiggans and Nonnenmacher (1999) provide evidence that in the two decades following the War, New York's trade grew significantly, while other ports stayed at roughly their 1811 level. The data they collected suggests that this growth is due primarily to the change in the law regarding auctions of imports, which discourages the setting of reservation prices. Both examples lend support to the theoretical prediction of our stylized model probably because they picture scenarios in which prospective buyers need a close scrutiny of the object to form their valuation as is assumed in our model.

Similar result concerning the optimal auction in the monopoly case, in which bidders can either exercise an outside option or enter the auction market have been derived in Engelbrecht-Wiggans (1987), McAffee and McMillan (1987*b*) and Engelbrecht-Wiggans (1993).

## Acknowledgements

I would especially like to thank my PhD thesis advisor Jürgen Eichberger for his guidance, encouragement and constant support. I am also very indebted to Roberto Burguet, Hans Gersbach, Hans Haller, Angel Hernando-Veciana, Dan Kovenock and Preston McAfee for valuable suggestions and insightful discussions.

# Appendix A

***Proof of lemma 2:***

We have to show that the function

$$R^i_j\Big((\underbrace{C,C,\ldots,C}_{I});m\Big)=\sum_{n=1}^{I}\binom{I-1}{n-1}m^{n-1}(1-m)^{I-n}\cdot B_n-C$$

is monotonically decreasing in $m$. Let us denote

$$G^{(l)}(m):=\binom{I-1}{l-1}m^{l-1}(1-m)^{I-l}.$$

We will first show that the functions

$$G_n(m):=\sum_{l=1}^{n}G^{(l)}(m)$$

are strictly monotonically decreasing in $m$ for $m\in[0,1]$ and $n\in\{1\ldots,I-2\}$. We have

$$\frac{dG^{(l)}(m)}{dm}=\binom{I-1}{l-1}\cdot m^l\cdot(1-m)^{I-l-2}\cdot\big[l\cdot(1+2\cdot m)-(I-1)\cdot m\big]$$

Thus $\frac{dG^{(l)}(m)}{dm}\gtrless 0$ for $l\gtrless\frac{(I-1)\cdot m}{1+2\cdot m}$. Let $\bar{l}$ be the highest integer, which is not larger than $\frac{(I-1)\cdot m}{1+2\cdot m}$. Then for all $n\in\{1,\ldots,\bar{l}\}$ the the functions $G_{I-1}(m)$ are obviously monotonically decreasing. For $n\in\{\bar{l}+1,\ldots,I-2\}$ observe that since

$$G_{I-1}(m)\equiv 1$$

one obtains

$$G_n(m)=1-\sum_{l=n+1}^{I-1}G^{(l)}(m)$$

Since for these $l$ we obtained $\frac{dG^{(l)}(m)}{dm} > 0$ it follows again that $G_{I-1}(m)$ are monotonically decreasing. The desired result follows from the inequalities

$$
\begin{aligned}
& B_1 \cdot \frac{d}{dm}\big(G^{(0)}(m)\big) + B_2 \cdot \frac{d}{dm}\big(G^{(1)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
& B_2 \cdot \frac{d}{dm}\big(G^{(0)}(m)\big) + B_2 \cdot \frac{d}{dm}\big(G^{(1)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) = \\
& B_2 \cdot \frac{d}{dm}\big(G_1(m)\big) + B_3 \cdot \frac{d}{dm}\big(G^{(2)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
& B_3 \cdot \frac{d}{dm}\big(G_2(m)\big) + B_4 \cdot \frac{d}{dm}\big(G^{(3)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
& . \\
& . \\
& B_{l+1} \cdot \frac{d}{dm}\big(G_l(m)\big) + B_{l+1} \cdot \frac{d}{dm}\big(G^{(l)}(m)\big) + \cdots + B_I \cdot \frac{d}{dm}\big(G^{(I-1)}(m)\big) > \\
& . \\
& . \\
& > B_I \cdot \frac{d}{dm}\big(G_{I-1}(m)\big) = 0.
\end{aligned}
$$

***Proof of lemma 1:***

The seller $j$ solves the problem of choosing $(p^j, z^j)$ so as to maximize

$$\Pi_j\Big((p^j, z^j); m^*\Big) = \sum_{n=1}^{I} \binom{I}{n} (m^*)^n (1-m^*)^{I-n} \cdot \int_0^1 Z_j^{(n)}(x_i) dF(x_i)$$

subject to the constraint

$$
\begin{aligned}
R^* &= R_j^i\Big((p^j, z^j); m^*\Big) \\
&= \sum_{n=1}^{I} \binom{I-1}{n-1} (m^*)^{n-1} (1-m^*)^{I-n} \cdot \int_0^1 \Big(x_i \cdot P_j^{(n)}(x_i) - Z_j^{(n)}(x_i)\Big) dF(x_i).
\end{aligned}
$$

The constraint can be rewritten as

$$I \cdot m^* \cdot R^* = \sum_{n=1}^{I} \binom{I}{n} (m^*)^n (1-m^*)^{I-n} \cdot \int_0^1 \Big(x_i \cdot P_j^{(n)}(x_i) - Z_j^{(n)}(x_i)\Big) dF(x_i).$$

Using this observation the maximization problem becomes equivalent to maximizing the expression

$$\sum_{n=1}^{I} \binom{I}{n} (m^*)^n (1-m^*)^{I-n} \cdot \int_0^1 \Big(x_i \cdot P_j^{(n)}(x_i)\Big) dF(x_i) - I \cdot m^* \cdot R^*.$$

The expectation obviously takes a maximum if the expression

$$\int_0^1 \Big(x_i \cdot P_j^{(n)}(x_i)\Big) dF(x_i)$$

is maximized for every $n$. Since for $x_{-i} \in [0,1]^{n-1}$ we have

$$P_j^{(n)}(x_i) = \int\limits_{[0,1]^{n-1}} p_i(x_i, x_{-i}) dF(x_{-i})$$

and we consider anonymous mechanisms one obtains for $x \in [0,1]^n$

$$\int_0^1 \Big( x_i \cdot P_i^{(n)}(x_i) \Big) dF(x_i) = \frac{1}{n} \cdot \int\limits_{[0,1]^n} \Big( \sum_{i=1}^{n} x_i \cdot p_i(x) \Big) dF(x).$$

The expression takes a maximum if for every participant $i$ the probability $p_i(x)$ is chosen so that

$$p_i(x) = \begin{cases} 1 & \text{if } x_i \text{ is the highest valuation,} \\ 0 & \text{otherwise.} \end{cases}$$

# Appendix B

***Proof of theorem 3:***

Consider the function

$$\begin{aligned} \overline{\varphi}(C) =& \frac{\partial}{\partial \overline{m}} \left( \sum_{n=1}^{I} \binom{I}{n} \overline{m}^n (1-\overline{m})^{I-n} \cdot S_n \right) \cdot \left. \frac{\partial \overline{m}(C, C^j)}{\partial C_j} \right|_{C^j = C} \\ &+ I \cdot \overline{m}(C, C) \\ &+ I \cdot \left. \frac{\partial \overline{m}(C, C^j)}{\partial C_j} \right|_{C^j = C} \cdot C. \end{aligned}$$

Observe that the first two terms are constants. Indeed, from the equation defining $\overline{m}(C, C^j)$ follows that $\overline{m}(C, C) = 1/J$ and that $\left. \frac{\partial \overline{m}(\cdot)}{\partial C_j} \right|_{C^j = C}$ is negative and constant with respect to $C$. The function in the last term is linear and decreasing in $C$. It follows that there exists a unique $\overline{C}$ for which $\overline{\varphi}(\overline{C}) = 0$. Since it is assumed that $\Pi_j \Big( C^j; \overline{m}(C, C^j) \Big)$ is concave in $C^j$ it follows that $C^j = \overline{C}$ is the unique (global) maximizer of this function. If $\beta \leq R_{(1/J)} - \overline{C}$, then all bidders find it optimal to enter the market with a probability of one and to visit each seller with a probability of $1/J$. The equilibrium participation fee in this case is $\overline{C}$.

The function $\underline{\varphi}(C^j, \beta)$ is also decreasing in $C^j$ because by assumption the function $\Pi_j \Big( C^j; \underline{m}(\beta, C^j) \Big)$ is concave. The unique maximizer of this function is $\underline{C}(\beta)$. If the entrance fee is $R_{(1/J)} - \beta$ and bidders enter the market with a probability of one, then their expected payoff is $\beta$. If the entrance fee is $\underline{C}(\beta) > R_{(1/J)} - \beta$, bidders exercise the outside option with positive probability. In this case $\underline{C}(\beta)$ is the equilibrium entrance fee.

If $\underline{C}(\beta) \leq R_{(1/J)} - \beta$, then observe that for $C^j < R_{(1/J)} - \beta$ bidders do not exercise the outside option and one obtains

$$\frac{\partial \Pi_j\Big(C^j; \overline{m}(C^j, R_{(1/J)} - \beta)\Big)}{\partial C^j}\Bigg|_{C^j < R_{(1/J)}-\beta} > \frac{\partial \Pi_j\Big(C^j; \overline{m}(C^j, R_{(1/J)} - \beta)\Big)}{\partial C^j}\Bigg|_{C^j = R_{(1/J)}-\beta}$$

$$> \frac{\partial \Pi_j\Big(C^j; \overline{C}\Big)}{\partial C^j}\Bigg|_{C^j = \overline{C}} = 0.$$

The first inequality applies due to the concavity of $\Pi_j\Big(\cdot;\cdot\Big)$ in $C^j$. The second inequality applies because (as we showed) $\overline{\varphi}(C) = \frac{\partial \Pi_j\left(C^j;C\right)}{\partial C^j}\Big|_{C^j=C}$ is decreasing.

For $C^j > R_{(1/J)} - \beta \geq \underline{C}(\beta)$ bidders exercise the outside option with positive probability. This is the case because every bidder will be indifferent between entering the market and exercising the outside option, if each seller uses the fee $R_{(1/J)} - \beta$, and all other bidders enter with a probability of one. One obtains

$$\frac{\partial \Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)}{\partial C^j}\Bigg|_{C^j > R_{(1/J)}-\beta} < \frac{\partial \Pi_j\Big(C^j; \underline{m}(\beta, C^j)\Big)}{\partial C^j}\Bigg|_{C^j = \underline{C}(\beta)} = 0.$$

In this case $R_{(1/J)} - \beta$ is the equilibrium participation fee.

***Proof of theorem 4:***

*Case* $I = 2$.

We have

$$\Pi_j\Big(C^j; C\Big) = m^2 \cdot S_2 + 2 \cdot m \cdot (1-m) S_1 + 2 \cdot m \cdot C_j,$$

where $m$ solves the equation

$$m \cdot B_2 + (1-m) \cdot B_1 - C_j = m \cdot B_1 + (1-m) \cdot B_2 - C.$$

Showing that this function is concave in $C_j$ is equivalent to show that the function is concave in $m$. After rearranging terms we obtain

$$\Pi_j\Big(m; C\Big) = m^2 \cdot (S_2 + 2B_2 - 2B_1) + m(2S_1 + B_1 - B_2 + C).$$

Further we make use of the following lemma.

**Lemma 3.** *For any probability distribution $F$ the equality $B_1 = B_2 + S_2$ is satisfied.*

*Proof.* A straightforward but somewhat cumbersome proof of this statement would be to substitute for $B_1, B_2$ and $S_2$ with the respective expressions defining these variables and check the equality. Here we offer a more intuitive argument. Let bidders 1 and 2 have the valuations $x_1$ and $x_2$. If bidder 1 participates alone in a second price auction with a zero reserve price, his payoff would be $x_1$. If he participates with bidder 2 then his payoff would be 0 if $x_1 < x_2$ and $x_1 - x_2$ otherwise. In the former case the seller's payoff is $x_1$ and in the latter case $x_2$. In both cases the sum of the buyers' and seller's payoff is $x_1$ just as in the case in which bidder 1 participates alone in a second price auction. Since this argument is valid for any $x_1$ and $x_2$ the claim follows. □

Applying this lemma it is now easy to see that $S_2 + 2B_2 - 2B_1 = B_2 - B_1 < 0$. The function is concave. The proof is analogous for the function

$$\Pi_j\Big(C^j;\beta\Big) = m^2 \cdot S_2 + 2 \cdot m \cdot (1-m)S_1 + 2 \cdot m \cdot C_j,$$

for which $m$ solves the equation

$$m \cdot B_2 + (1-m) \cdot B_1 - C_j = \beta.$$

*Case* $I = 3$.
We have

$$\Pi_j\Big(C^j;C\Big) = m^3 \cdot S_3 + 3m^2(1-m) \cdot S_2 + 3 \cdot m \cdot (1-m)^2 S_1 + 3 \cdot m \cdot C_j,$$

where $m$ solves the equation

$$m^2 \cdot B_3 + 2m(1-m) \cdot B_2 + (1-m)^2 \cdot B_1 - C_j = (1-m)^2 \cdot B_3 + 2m(1-m) \cdot B_2 + m^2 \cdot B_1 - C.$$

After solving the latter equation and substituting in the former one, we obtain

$$\Pi_j\Big(m;C\Big) = m^3 \cdot (S_3 - 3S_2) + m^2 \cdot (3S_2 - 6B_1 + 6B_2) + m \cdot (\text{term}) + (\text{another term}).$$

For the second derivative with respect to $m$ we obtain

$$\frac{\partial^2 \Pi_j\Big(m;C\Big)}{\partial^2 m} = 6m(S_3 - 3S_2) + 2(3S_2 - 6B_1 + 6B_3).$$

It is clear that $S_3 - 3S_2 < 0$. Further

$$3S_2 - 6B_1 + 6B_2 = 3S_2 - 3S_2 - 3B_2 - 3B_1 + 6B_3 = -3B_2 - 3B_1 + 6B_3 < 0.$$

It follows that the function is concave. Analogously for the function

$$\Pi_j\Big(C^j;\beta\Big) = m^3 \cdot (S_3 + 3C) + 3m^2(1-m)(S_2 + 2C) + 3m(1-m)^2(S_1 + C),$$

where

$$m^2 B_3 + 2m(1-m)B_2 + (1-m)^2 B_1 - C = \beta$$

we obtain for the second derivative

$$\frac{\partial^2 \Pi_j\Big(m;\beta\Big)}{\partial^2 m} = 6m(S_3 + 3B_3 - 3B_2) - 6S_2 < 6S_3 - 6S_2 < 0.$$

***Proof of theorem 5:***

For the uniform distribution it is easy to show that

$$B_n = \frac{1}{n(n+1)}; S_n = \frac{n-1}{n+1}.$$

Then the function

$$\Pi_j\Big(C^j;\beta\Big) = \sum_{n=1}^{I} \binom{I}{n} \overline{m}^n (1-\overline{m})^{I-n} \cdot S_n + I \cdot m \cdot C_j,$$

where

$$R_{(m)} - C_j = \beta$$

should be shown to be concave. Again after solving the last equality and substituting in the former one, one obtains

$$\begin{aligned}
\Pi_j\Big(m;\beta\Big) &= \sum_{n=1}^{I} \binom{I}{n} m^n (1-m)^{I-n} \cdot \frac{n-1}{n+1} \\
&\quad + I \cdot \sum_{n=1}^{I} \binom{I-1}{n-1} \frac{m^n (1-m)^{I-n}}{n(n+1)} - \beta I m = \\
&= \sum_{n=1}^{I} \binom{I}{n} \frac{m^n (1-m)^{I-n}}{(n+1)} [n-1+1] - \beta I m \\
&= \sum_{n=1}^{I} \binom{I}{n} m^n (1-m)^{I-n} \cdot \frac{n}{n+1} - \beta I m \\
&= \underbrace{\sum_{n=1}^{I} \binom{I}{n} m^n (1-m)^{I-n}}_{(*)} - \underbrace{\sum_{n=1}^{I} \binom{I}{n} \frac{m^n (1-m)^{I-n}}{n+1}}_{(**)} - \beta I m.
\end{aligned}$$

Observe that

$$\begin{aligned}
(*) &= [m + (1-m)]^I - 1 \cdot m^0 (1-m)^I = 1 - (1-m)^I, \\
(**) &= \frac{1}{(I+1) \cdot m} \sum_{n=1}^{I} \frac{I!(I+1)}{n!(I-n)!(n+1)} m^n (1-m)^{I-n} \\
&= \frac{1}{(I+1)m} \sum_{n=1}^{I} \binom{I+1}{n+1} m^{n+1} (1-m)^{I-n} \\
&= \frac{1}{m(I+1)} - \frac{(1-m)^I + 1}{m(I+1)} - (1-m)^I.
\end{aligned}$$

Substituting $(*)$ and $(**)$ in the previous equation one obtains

$$\begin{aligned}\Pi_j\Big(m;\beta\Big) &= 1-(1-m)^I - \frac{1}{m(I+1)} + \frac{(1-m)^{I+1}}{m(I+1)} + (1-m)^I - mI\beta \\ &= 1 + \frac{[(1-m)^{I+1}-1]}{m(I+1)} - mI\beta \\ &= 1 + \frac{(-m)[(1-m)^I + (1-m)^{I-1} + \cdots + (1-m) + 1]}{m(I+1)} - mI\beta \\ &= 1 - \frac{1}{I+1}\sum_{n=0}^{I}(1-m)^n - mI\beta.\end{aligned}$$

For the first derivative one obtains

$$\frac{\partial \Pi_j\Big(m;\beta\Big)}{\partial m} = \frac{1}{I+1}\sum_{n=1}^{I} n(1-m)^{n-1} - I\beta$$

and for the second

$$\frac{\partial^2 \Pi_j\Big(m;\beta\Big)}{\partial^2 m} = -\frac{1}{I+1}\sum_{n=2}^{I} n(n-1)(1-m)^{n-2} \leq 0.$$

The concavity of function $\Pi_j\Big(C^j;\overline{m}(C,C^j)\Big)$ is difficult to show analytically for an arbitrary number of buyers and sellers even for $F$ uniformly distributed. It appears however that this property holds. For all $J \in \{2,3,\ldots,100\}$ and $I \in \{2,3,\ldots,100\}$ we computed using a C++ program the second derivative and established that it is negative at all discrete points between 0 and 1 with a step of $0,00001$. The source code is available from the author upon request.

***Proof of claim 1:***

If seller 1 charge a participation fee of $C^1$, seller 2 a participation fee of $C$ and bidders enter the market with a probability of one, then $\overline{m}(C,C^1)$ solves the equation

$$\begin{aligned} m\cdot B_2 + (1-m)\cdot B_1 - C^1 = (1-m)\cdot B_2 + m\cdot B_1 - C^2 \Leftrightarrow \\ \overline{m}(C,C^1) = \frac{(C^1-C)+B_2-B_1}{2\cdot(B_2-B_1)}.\end{aligned}$$

The expected payoff of seller 1 is

$$\Pi_1\Big(C^1;\overline{m}(C^1,C)\Big) = \overline{m}(C,C^1)^2\cdot(S_2+2C^1) + 2\overline{m}(C,C^1)(1-\overline{m}(C,C^1))(S_1+C^1).$$

The equation $\overline{\varphi}(C)=0$ has the solution $\overline{C} = B_1 - B_2 - S_2/2$. Recall that in lemma 3 we showed that $B_1 = B_2 + S_2$. Since the *ex ante* payoff of each bidder is $B_1/2 + B_2/2 - S_2/2 =$

$B_2$ and bidders will enter with a probability of one if $\beta < B_2$, the participation fee is $S_2/2$. For $j = 1, 2$ the function $\underline{m}(\beta, C^j)$ satisfies the equation

$$R^i_j\Big(C^j; m\Big) = \beta \Leftrightarrow$$
$$m \cdot B_2 + (1-m) \cdot B_1 - C^j = \beta \Leftrightarrow$$
$$\underline{m}(\beta, C^j) = \frac{\beta + C^j - B_1}{(B_2 - B_1)}.$$

The expected payoff of seller $j$ is

$$\Pi_j\Big((C^j, \beta; \underline{m}\Big) = \underline{m}^2 \cdot (S_2 + 2C^j) + 2\underline{m}(1-\underline{m})(S_1 + C^j).$$

The equation

$$\underline{\varphi}(C^j, \beta) = 0$$

has the solution $\underline{C} = (\beta - B_1)(B_1 - B_2 - S_2)/(S_2 - 2B_1 + 2B_2) = 0$.

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