Valuing Mortgage Insurance Using Bivariate Binomial Option-Pricing Technique

by

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Abstract

This study attempts to calculate mortgage insurance premium that a borrower is supposed to pay, to meet the losses of the lender when the borrower defaults. When the default occurs the payment by the insurer is the difference between the defaulted house's value and loan balance, up to a specified fraction of the loan balance. Because the default is an option that the borrower may exercise whenever he wishes to, it follows that the mortgage insurance is an option too, and the premium of it can be modeled and calculated as a price of that option. This study uses three-dimensional bivariate binomial options pricing technique to value mortgage insurance in a fixed-rate mortgage environment where the borrower has both prepayment and default options in hand. Finally we used Turkey data and applied our model to this market and calculate the mortgage insurance premium of a possible mortgage market which is supposed to be established soon.

Key Words and Phrases: Mortgage insurance, option-based pricing, binomial tree

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1 Introduction

Long term housing loans are associated with many risks: default risk, credit risk etc... Default risk, which is the borrowers' possibility to fail to pay their debt, is one of the main anxieties of the lenders who originate and hold mortgage loans. Even though the lenders has the house as a collateral and hence the right to foreclosure as a guarantee, this process is very costly and time consuming to them considering legal expenses, costs to resell the property etc...As a result the issuers of these loans (lenders) ask for mortgage insurance against default to protect themselves against credit risk.

The mortgage insurance is different from other type of insurances in many senses. Dennis et al (1997), mention these differences: First, mortgage insurance contracts have multi-period coverage. Second, the termination date is known ex-ante in contrast to for example life insurances and the risk decreases rather than increases. Third, it includes high systematic risk because the default rates, and the prepayment rates are highly dependent on macroeconomic variables such as interest rates. And finally when the insurance is obligatory, even though it is the borrower who pays the premium the insurance covers the risk to the lender.

Concerning these differences, it is highly important to value the appropriate price of the mortgage insurance with the proper tool. Kau et al (1993) mention the bad experiences in USA resulting with serious losses of big insurance firms such as Ticor Mortgage Insurance of Los Angeles (with an exposure of $166 million), Republic Mortgage Insurance (with an exposure of $66 million), and Mortgage Guaranty Insurance Corporation (with an exposure of $15 million). To avoid these losses companies should be careful of their pricing methodology. There are different types of valuing mortgage insurance proposed by economists and used in many countries. For example in the

United States, Fannie Mae and Freddie Mac charge borrowers a fixed premium rate about 25 basis points (0.25%) of the remaining balance. Dennis et al (1997) claim that this type of premium structure is not efficient since it doesn't match the expected net present value of possible potential losses that can be faced by the lenders to the value of premium income. The reason is that generally it is the beginning years during which the lenders face the highest probability of default risk comparing to the other years.

A more appropriate model for pricing should concern two crucial properties of mortgages: prepayment and default. When these two properties are modeled as call and put options respectively, it is easy to match the expected net present value of possible losses (We explain this in section 2 in detail). This leads the researchers to use option[1] pricing techniques to value the premium of the mortgage insurance. Some studies focus on the right to prepay a mortgage, but ignore the possibility of default (Dunn and McConnell (1981),Buser and Hendershott (1983) and Hall (1985)) , some of them do it the other way around (Cunningham and Hendershott (1984)). On the other hand Kau et al (1992) provide a model with both prepayment and default opportunities being present. Deng et al (2000) state that the option model does a good job of explaining default and prepayment, but also add that the simultaneity of the options is also very important in explaining the behavior. Hilliard et al (1998) use a bivariate binomial option pricing technique to value prepayment and default options in a fixed-rate mortgages, and use interest rate and real estate value as two state variables. Bardhan et al (2003), hereafter BKU, use Black and Scholes formula to calculate the insurance premium. They take the loss of the insurer as an option,

[1]Option is a financial instrument which gives the holder the right to buy (call option) or sell (put option) a certain asset for an exercise price determined now.

state two different types of contracts between borrower and insurer, and come up with the premium that the borrower should pay for each type of contract. BKU apply their idea to the case of Serbia and calculate an insurance premium.

In this paper, we price insurance of a fixed-rate mortgage where the borrower has both default and prepayment opportunities. If the borrower defaults he loses the option to prepay; and similarly if he prepays he loses the option to default. When the borrower defaults, the insurer is supposed to pay the difference between the defaulted house's value and loan balance, up to a specified fraction of the loan balance. We formulate this loss of the insurer as the difference of two options, and try to calculate the premium as the price of the insurance option. The model is a bivarite binomial option pricing technique, which is the three dimensional version of the Binomial tree, proposed by Hilliard et al (1996), hereafter HST (1996). The advantage of this model is that it allows us to use two underlying state variables that can be correlated. Because we believe that it is not only the house prices but also the interest rate that derives the strategic movement of the option holder, this model better fits our model than the Binomial tree. The dynamics of the model proceeds as follows. First, we transform the two stochastic processes to processes with constant variance so that binomial grids recombine. Then another transformation is made to make two once-transformed processes uncorrelated. Thus after two transformations we obtain uncorrelated and constant-variance processes. So that we can combine them through four joint probabilities which are the different combination of up and down movements of two state variables. The procedure takes place in a three dimensional Binomial tree, such that for every node we have four branches stemming. We define our contract and use this tree to calculate the premium of the insurance (as if it were the price of an option) and state our results. Finally we compare our results with Erdem (2004).

The paper is organized as follows: Section 2 briefly describes the mortgage framework, its characteristics and mortgage insurance scheme. Section 3 states the details of the bivariate binomial option pricing technique.Section 4 gives the calibration results while Section 5 makes comparison with previous works made using different techniques. And finally section 6 concludes

2 Mortgage Framework

It is assumed that the mortgage loan is a simple amortizing loan with a fixed monthly contract rate c (annual contract rate/12), and it is paid off on monthly basis for T months. The characterization of the loan and the payments are as follows.

That is the payment, x, of the borrower each month is fixed and equal to

$$\begin{aligned} B_0 &= \frac{x}{1+c} + \frac{x}{(1+c)^2} + \frac{x}{(1+c)^3} + ... + \frac{x}{(1+c)^T} \\ B_0 &= x\frac{(1+c)^T - 1}{(1+c)^T c} \end{aligned}$$

Thus

$$x = B_0 \left[\frac{(1+c)^T c}{(1+c)^T - 1} \right] \tag{1}$$

where B_0 is the initial loan amount, T is the terms (number of months) of the loan.

Mortgage balance (unpaid), B_t, at time t (after the t^{th} payment) is

$$
\begin{aligned}
B_t &= \frac{x}{1+c} + \frac{x}{(1+c)^2} + \frac{x}{(1+c)^3} + ... + \frac{x}{(1+c)^{T-t}} \\
&= x\left[\frac{(\frac{1}{1+c})^{T-t+1} - (\frac{1}{1+c})}{(\frac{1}{1+c}) - 1}\right] \\
&= x\left[\frac{(1+c)^{T-t} - 1}{c(1+c)^{T-t}}\right] \qquad (2)
\end{aligned}
$$

Substituting x from (1) yields

$$
B_t = B_0\left[\frac{(1+c)^T - (1+c)^t}{(1+c)^T - 1}\right] \qquad (3)
$$

The model lets the interest rates change in the binomial tree to up and down values. That is why the present value of the reamaing loan balance, PV_t, is not equal to mortgage balance B_t. The present value of the reamining mortgage balance is

$$
\begin{aligned}
PV_t &= \frac{x}{1+r} + \frac{x}{(1+r)^2} + \frac{x}{(1+r)^3} + ... + \frac{x}{(1+r)^{T-t}} \\
&= x\left[\frac{(\frac{1}{1+r})^{T-t+1} - (\frac{1}{1+r})}{(\frac{1}{1+r}) - 1}\right] \\
&= x\left[\frac{(1+r)^{T-t} - 1}{r(1+r)^{T-t}}\right] \qquad (4)
\end{aligned}
$$

Note that if the interest rate, r, were fixed and equal to initial contract rate c, the equations (2) and (4) would be identical. But since we allow interest rates to change, we have different r values for different t.

2.1 Default and Prepayment

Throughout the life of the mortgage, the mortgage holder may default or prepay her loan depending on the financial environment. Here we do not take into account non-financial reasons while we are dealing with the mortgage holders decisions. Financial reasons include the movements of the interest rates, and the house prices. The important point is the intereaction of default and prepayment options since using one of these options excludes the other. That is why they cannot be thought as two seperate options. Let's examine both cases:

1.Default: The borrower defaults when the value of her house H_t becomes less than her remaining mortgage balance B_{t-1}[2](e.g. for a sufficiently low value of house it is not worth to continue). If the borrower defaults at the beginning of any time t, this means that she exercises her default option, D_t, and she rejects to pay her mortgage balance B_{t-1}. Since she loses the oppurtunity to prepay, the value of the prepayment option, P_t, becomes zero. In the case of default the insurer must pay her possible losses according to the insurance model that is defined in Section 2.2.

The value of the default option is calculated as follows:

$$\begin{aligned} D_t &= \max(B_{t-1} - \widetilde{H}_t, 0) \\ P_t &= 0 \end{aligned}$$

Note that default is a put option.

2.Prepayment: The borrower prepays her mortgage loan when the interest rate

[2] Even though B_t stands for the mortgage balance at time t, it should be noted that it is calculated assuming that the time t payment is made. That is why an agent who will decide whether or not to make the payment concerns about B_{t-1}.

falls, that is when the present value of her mortgage loan balance PV_t exceeds her unpaid mortgage balance B_{t-1}. If the borrower prepays at the beginning of time t, this means that she exercises her prepayment option, P_t, and loses the oppurtunity to default. Then the value of the default option, D_t, becomes zero.

The value of the prepayment option is:

$$\begin{aligned} P_t &= \max(\widetilde{PV} - B_{t-1}, 0) \\ D_t &= 0 \end{aligned}$$

where $\widetilde{PV}$ is the present value of the future payments discounted with the current interest. Note that prepayment is a call option.

The borrower decides which option to exercise according to the their values, i.e which one gives her the highest payoff.

2.2 Insurance Model

In this section we will state explicitly the model we are using as an insurance contract.

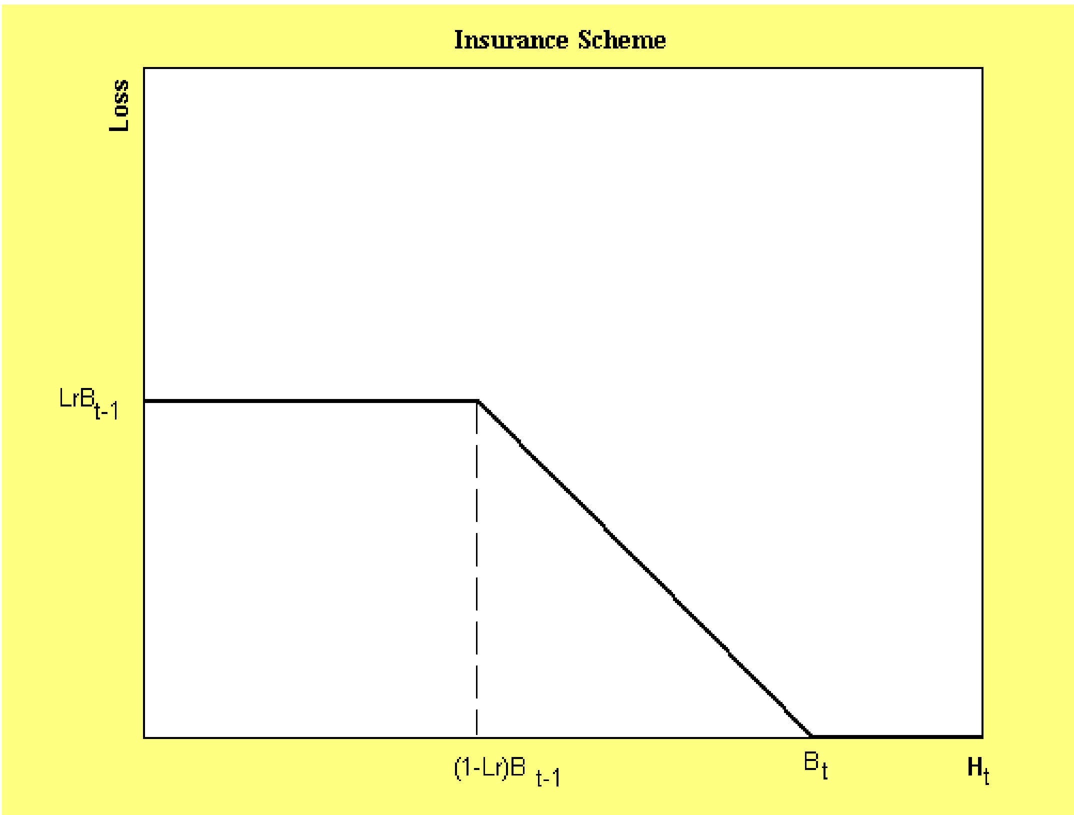
Suppose that a borrower defaults at time t. The amount the lender will receive from the insurance agency depends on the shape of the contract. We define the insurance company's loss function at the time of default as follows:

$$Loss_t = \max(0, \min(B_{t-1} - H_t, L_R B_{t-1})) \tag{5}$$

where L_R is the loss ratio defined as the percentage of the loss that is covered by insurance company.

Equation (5) says that the insurer pays the difference between the defaulted house's value H_t, and loan balance B_{t-1}, up to a specified fraction L_R of the loan balance.

Thus, the maximum loss that the insurance agency may incur is $L_R B_{t-1}$ in this Scheme:



Note that this loss function can be replicated through simple portfolios of put options[3]. In such an option H_t can be thought of as an underlying asset. Thus, for

[3] An American put option gives the holder the right to sell an underlying asset for a exercise price K at any time until the maturity. The payoff to the holder is $max(K - S_T, 0)$ where S_T is the final price of the underling asset.

the Two-Option Scheme, the expression for (5) is equal to the following expression:

$$Loss_t = \max(K_1 - \widetilde{H}_t, 0) - \max, (K_2 - \widetilde{H}_t, 0) \tag{6}$$

where

$$K_1 = B_{t-1} \text{ and } K_2 = (1 - L_R)B_{t-1}$$

Note also that this expression is equivalent to a payoff of a portfolio of a long position in a put option with strike price $K_1 = B_{t-1}$ and a short position in a put with strike price $K_2 = (1 - L_R)B_{t-1}$.

So expected value of the insurer's loss (L) is given by the following discounted expression:

$$L = Put(K_1, t) - Put(K_2, t) \tag{7}$$

3 Bivariate Binomial Model

In this study, we try to examine the behaviour of the mortgage holder's decisions which depend on the financial variables, namely house prices, and interest rates(XXX). But these two variables may very well be correlated. That is why their interaction through time should de concerned in the model. The bivariate binomial model is a good tool for this. This model lets two variables make up and down movements while on the other hand the variables have the freedom to be correlated. In this section we are going to describe the stochastic processes of the underlying state variables, namely the house prices and interest rates, and state the bivariate binomial options pricing methodology, following Hilliard et al (1996).

3.1 Transformation of Original Processes to Constant Variance Processes

Let the house prices follow the risk-adjusted processes given by:

$$dH/H = m\ dt + \sigma_H(t)dZ_H \tag{8}$$

and interest rates given by

$$dr = a\ dt + bdZ_r \tag{9}$$

where $m = r(t)$, $a = [\kappa(\theta - r(t)) - \lambda b]$, λ is the market price of risk. dZ_H and dZ_r are Wiener processes with $E[dZ_H] = 0$, $E[dZ_r] = 0$, $dZ_H^2 = dt$, $dZ_r^2 = dt$ and related with the instantenous correlation coefficient ρ such as $dZ_H dZ_r = \rho dt$. Assume that $\kappa, \theta, \sigma_H, \lambda, b$ and ρ are constants.

In (9), the interest rate follows a mean-reverting process, that is the drift $\kappa(\theta - r(t))$ pulls the process back to the long term mean θ, at a rate κ.

The variable H is the house prices, and r is the short-term interest rates. Consistent with Nelson and Ramaswamy (1990) , nearly any stochastic process can be transformed into a constant variance process and therefore be modeled as a recombining binomial.

To transform (8) to a normal process, let $S = ln(H)$, and use Ito's rule, so that

$$dS = (\frac{1}{H} mH - \frac{H^2\sigma_H^2}{2H^2})dt + \frac{1}{H}\sigma_H\ H\ dZ_H$$

$$dS = m^* dt + \sigma_H \; dZ_H \tag{10}$$

where $m^* = [m - \sigma_H^2/2]$. Next, transform to the process ϕ with unit volatility requiring

$$\phi = S/\sigma_H \tag{11}$$

Again using Ito's rule, equation (11) implies the following result:

$$d\phi = \left(\frac{m^*}{\sigma_H} - \frac{S\sigma'_H}{\sigma_H^2} \right) dt + \; dZ_H \equiv \mu dt + dZ_H \tag{12}$$

where

$$\sigma'_H = \frac{\partial \sigma_H}{\partial t} \text{ and } \mu = \frac{m^*}{\sigma_H} - \frac{S\sigma'_H}{\sigma_H^2}$$

Equation (12) represents the original equation (8) transformed to yield unit variance.

Since dr is a constant variance process, we do not need to transform anything, and use (9) instead. Just to keep the consistency with the literature let

$$R = r$$

so that

$$dR = a \; dt + bdZ_r \tag{13}$$

with a defined with (9)

3.2 Orthogoanlizing the Once-transformed Processes

To ensure zero covariance, we jointly transform equations (12) and (13) as follows:

$$
\begin{aligned}
X_1 &= b\phi + R \qquad (14)\\
X_2 &= b\phi - R
\end{aligned}
$$

so that

$$
\begin{aligned}
dX_1 &= bd\phi + dR \qquad (15)\\
dX_2 &= bd\phi - dR
\end{aligned}
$$

Notice that the twice-transformed processes given by (15) have zero covariance since[4]

[4]Note the multiplication rules for a hypothetical stochastic differenetial $d\gamma = \mu dt + \sigma dZ$:

*	dZ	dt
dZ	dt	0
dt	0	0

$$\begin{aligned} cov(dX_1, dX_2) &= cov(bd\phi + dR, bd\phi - dR) \\ &= cov(bd\phi, bd\phi) - cov(bd\phi, dR) + cov(dR, bd\phi) - cov(dR, dR) \\ &= b^2 var(d\phi) - var(dR) \\ &= b^2(dt) - b^2\rho + b^2\rho - (b^2 dt) \\ &= 0 \end{aligned} \tag{16}$$

Expanding (15) gives uncorrelated and constant volatility processes to be modeled in the bivariate binomial setup:

$$dX_1 = bd\phi + dR = b(\mu dt + dZ_H) + (adt + bdZ_r) = (b\mu + a)dt + b(dZ_H + dZ_r) \tag{17}$$

$$dX_1 = bd\phi - dR = b(\mu dt + dZ_H) - (adt + bdZ_r) = (b\mu - a)dt + b(dZ_H - dZ_r) \tag{18}$$

The equations (17) and (18) can be rewritten as[5]:

$$dX_1 = (b\mu + a)dt + b\sqrt{2(1+\rho)}dz_1 \equiv \mu_1 dt + \sigma_1 dz_1 \tag{10b}$$

$$dX_2 = (b\mu - a)dt - b\sqrt{2(1-\rho)}dz_2 \equiv \mu_2 dt - \sigma_2 dz_2 \tag{11b}$$

where

$$\mu_1 = (b\mu + a), \sigma_1 = b\sqrt{2(1+\rho)}$$

[5] The derivation is in the Appendix A.

and

$$\mu_2 = (b\mu - a), \sigma_2 = b\sqrt{2(1-\rho)}$$

The solution to the option pricing problem involves modeling these transformed processes((17) and (18)) as two seperate binomial processes and then combining the two binomial trees via four joint probabilities.

3.3 Ensuring Weak Convergence and Legitimate Probabilities

To calculate the mean of the discrete S process,consider the following definition: the probability of an up-jump in X_1 is

$$p = 0.5 + \mu_1\sqrt{\Delta t}/2\sigma_1 \tag{19}$$

Similarly, the probability of an up-jump in X_2 is q.

Figure 1 shows how we construct a three-dimensional branching.

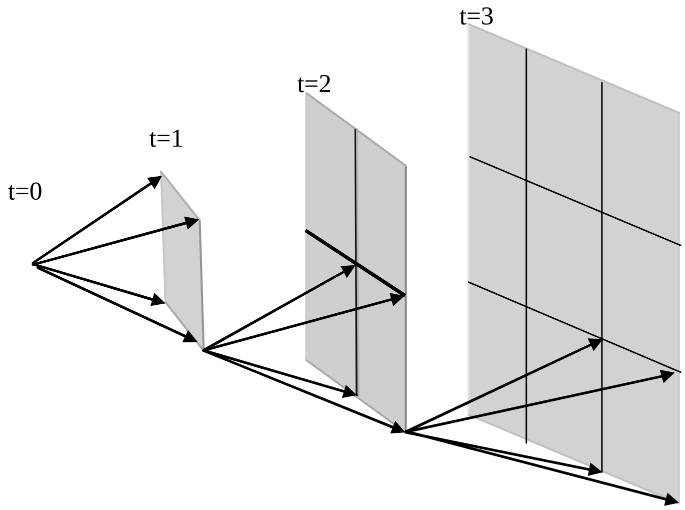


FIGURE 1: There are four branches stemming from every node. At time t, we have (t+1)*(t+1) nodes. Note not all the nodes are shown at t=2,3.

By independence of X_1 and X_2, define joint probabilities as:

$$
\begin{aligned}
P_{11} &= prob(X_1 + \sigma_1\sqrt{\Delta t}, X_2 - \sigma_2\sqrt{\Delta t}) = p(1-q) \\
P_{12} &= prob(X_1 + \sigma_1\sqrt{\Delta t}, X_2 + \sigma_2\sqrt{\Delta t}) = pq \\
P_{21} &= prob(X_1 - \sigma_1\sqrt{\Delta t}, X_2 - \sigma_2\sqrt{\Delta t}) = (1-p)(1-q) \\
P_{22} &= prob(X_1 - \sigma_1\sqrt{\Delta t}, X_2 + \sigma_2\sqrt{\Delta t}) = (1-p)q
\end{aligned}
$$

Further, the marjinal probability of an up-jump in X_1 is $p = P_{11} + P_{12}$, and the probability of an up-jump in X_2 is $q = P_{21} + P_{22}$. By allowing multiple jumps in X_1 and X_2, HST (1996) calculates p as

$$p = 1/2 - k + \frac{\mu\sqrt{\Delta t}}{2\sigma}$$

3.4 Pricing Mechanics

1. Compute $\mu_1, \mu_2, \sigma_1, \sigma_2$. The expanded versions are in the appendix A.

2. From step 1, compute X_1^+, X_1^-, X_2^+, and X_2^-. For instance, the new up-jump value X_1^+ would be given by

$$X_1^+ = X_1 + (2k_1 + 1)\sigma_1\sqrt{\Delta t}$$

   where X_1 is the old node value. Since Because that $k_1 = 0$ almost always satisfies...., we take the above equation as

$$X_1^+ = X_1 + \sigma_1\sqrt{\Delta t}$$

3. At each node compute the accompanying risk-neutralized probabilities for each possible outcome:

$$\begin{aligned}
X_1^+, X_2^- &: \quad P_{11} = p(1-q) \\
X_1^+, X_2^+ &: \quad P_{12} = pq \\
X_1^-, X_2^- &: \quad P_{21} = (1-p)(1-q) \\
X_1^-, X_2^+ &: \quad P_{22} = (1-p)q
\end{aligned}$$

4. Transform from X_1, X_2 back to H and r at each node as follows:

$$
\begin{aligned}
\phi &= \frac{X_1 + X_2}{2b} \\
R &= \frac{X_1 - X_2}{2} \\
r &= R = \frac{X_1 - X_2}{2} \\
S &= \phi \sigma_H \\
H &= e^S = e^{\left[\frac{X_1+X_2}{2b}\sigma_H\right]}
\end{aligned}
$$

5. Evaluate the derivative security defined on r and S, employing the usual backward deduction procedure. Check at each node for the possibility of early exercise.

4 The Calibration

4.1 The Data

We used Turkish Libor rates of the interbank market from September 2002 through May 2005 and used it as the interest rates.

Unfortunately National Bureau of Statistics (DIE) of Turkey does not have a house price index. That is why we needed to use some kind of proxy. Rent index which is a part of consumer price index could be a good candidate. Note that there is no upper or lower bans for the rent prices in Turkey. That is why we obtained rent index subitem from January 2002 through May 2005 used it as a proxy for the house prices.

4.2 Estimation of the Parameters

Rewriting the equations (8) and (9) as:

$$dH/H = m\ dt + \sigma_H dZ_H \tag{20}$$

$$dr = \left[\kappa\left(\theta - \frac{\lambda b}{\kappa} - r(t)\right)\right]\ dt + bdZ_r \tag{21}$$

Using maximum likelihood (ML) method, we first estimate the parameters for house price process, (20), as[6]:

$$\begin{aligned} \widehat{m}_T &= \frac{1}{T}\sum_{t=1}^{T}\Delta \log H_t + \frac{\widehat{s}_T^2}{2} \\ \widehat{\sigma}_H &= \widehat{s}_T^2 \end{aligned} \tag{22}$$

where

$$\widehat{s}_T^2 = \frac{1}{T}\sum_{t=1}^{T}(\Delta \log H_t - \widehat{m}_T)^2$$

Now, we will estimate the parameteres for the short term interest rate process. Let $\theta^* = \theta - \frac{\lambda b}{\kappa}$ in the equation (21). This equation has a discrete time counterpart as [7]:

$$r_t = \theta^*\left[1 - e^{-\kappa}\right] + e^{-\kappa} r_{t-1} + b\left(\frac{1 - e^{-2\kappa}}{2\kappa}\right)^{1/2}\varepsilon_t$$

where ε_t is a standardized Gaussian white noise. This equation is a AR(1) (autore-

[6] For derivation look at Gourieroux and Jasiak (2001)

[7] For derivation look at Gourieroux and Jasiak (2001)

gressive process of order 1) process in disrete time. Thus it can be written as:

$$r_t = \theta^* \left[1 - \gamma\right] + \gamma r_{t-1} + \eta \varepsilon_t$$

where $\gamma = e^{-\kappa}$ and $\eta = b\left(\frac{1-e^{-2\kappa}}{2\kappa}\right)^{1/2}$. The ML estimators of the parameters θ^*, γ, η are asymptotically independent but unfortunately we have a unit root problem in our data. We can estimate the parameters of unit root processes by OLS and assess their asymptotic properties. The OLS estimators are

$$\widehat{\gamma} = \frac{\sum_{t=2}^{T}\left(r_t - \frac{1}{T-1}\sum_{t=2}^{T} r_t\right)\left(r_{t-1} - \frac{1}{T-1}\sum_{t=2}^{T} r_{t-1}\right)}{\sum_{t=2}^{T}\left(r_{t-1} - \frac{1}{T-1}\sum_{t=2}^{T} r_{t-1}\right)^2} \tag{23}$$

and

$$\widehat{\theta^* \left[1 - \gamma\right]} = \frac{1}{T-1}\sum_{t=2}^{T} r_t - \widehat{\gamma}\frac{1}{T-1}\sum_{t=2}^{T} r_{t-1}$$

Thus

$$\widehat{\theta}^* = \frac{1}{\left[1 - \widehat{\gamma}\right]}\left[\frac{1}{T-1}\sum_{t=2}^{T} r_t - \widehat{\gamma}\sum_{t=2}^{T} r_{t-1}\right] \tag{24}$$

$$\widehat{\eta}^2 = \frac{1}{T}\sum_{t=1}^{T} \widehat{\varepsilon}_t^2 \tag{25}$$

where the residuals are defined by $\widehat{\varepsilon}_t = r_t - \widehat{\theta}^* \left[1 - \widehat{\gamma}\right] - \widehat{\gamma} r_{t-1}$. From these ML estimators of the parameters γ, θ^*, η, we can infer the ML estimators of the parameters

κ, b

$$\begin{aligned} \widehat{\kappa}_T &= -\log \widehat{\gamma} \\ b^2 &= -\left(\frac{2\log \widehat{\gamma}}{1-\widehat{\gamma}^2}\right)\widehat{\eta}^2 \end{aligned}$$

Since it is impossible to simultaneously determine θ and λ from the equation $\theta^* = \theta - \frac{\lambda b}{\kappa}$, we use an emprical estimation of market price of risk, as $\widehat{\lambda}$= 0.20 and estimate θ as

$$\widehat{\theta} = \theta^* + \frac{\lambda b}{\kappa}$$

Table 1 gives the estiamated parameters of the model with Turkey data.

TABLE 1.**Estimated Parameters**		
Data	**Symbol**	**Estimation**
Mortgage Term	T	15 years
Contract Rate	c	15%
Spot Rate	r	20%
Spot Rate Volatility	b	0.88%
House Price Volatility	σ_H	8%
Correlation Coefficient	ρ	various
Steady State Spot Rate	θ^*	40%
Speed of Convergence	κ	0.3%
Market Price of Risk	λ	20%
Payment Fraction	$\mathbf{L}_R$	20%
Loan To Value Ratio	**LTV**	Various

4.3 Calibration

Using (7) we model our mortgage insurance premium as the difference of the price of two put options. Each put option is valued with the three-dimensional bivariate binomial tree described above. Table 2 gives the results with the estimated parameters in the Table 1 with different loan to value ratios, LTV.

TABLE 2. Simulation of the Insurance Premium

LTV	Prepayment Prem.	Default Prem.	Insurance Prem.
0.80	5.8543	0.00047	0.00041
0.85	6.2113	0.0052	0.0045
0.90	6.5195	0.0395	0.0330
0.95	6.6017	0.2215	0.18139
The results are expressed as a percentage of the initial house price.			
The parameters are $r = 20\%, c = 15\%$, $\sigma_H = 0.15, b = 0.08$,			
$\rho = -0.20$, $\theta^* = 40\%$, $\lambda = 20\%$, $\kappa = 0.3\%$, $\mathbf{L}_R = 0.20$.			

Note that the insurance premium is positively correlated with LTV, which is quite intuitive. The more is the LTV, the more is the loan of the borrower which requires more premium.

5 Comparison with Previous Works and Comparative Statics

5.1 Comparison with Previous Works

Here we compare our results with Erdem (2004) who calculated the same insurance premiums with Black and Scholes formula.

TABLE 3. Premiums with Black and Scholes Formula						
Premium Structure with different LTV Ratios						
LTV	μ	σ	r	c	L_R	Premium
70%	0.06	0.08	0.2	0.15	0.20	0%
80%	0.06	0.08	0.2	0.15	0.20	0,001%
90%	0.06	0.08	0.2	0.15	0.20	0,003%

If we compare Table 2 and Table 3 we see that the difference between two models is not more than %0.06.

5.2 Comparative Statics

Here we calibrate the model with $\rho = 0$.

TABLE 4. Simulation of the Insurance Premium with $\rho = 0$			
LTV	Prepayment Prem.	Default Prem.	Insurance Prem
0.80	5.8578	0.0017819	0.00145
0.85	6.1979	0.012698	0.0099
0.90	6.4338	0.075998	0.0579
0.95	6.3293	0.34573	0.2487
The results are expressed as a percentage of the initial house price.			
The parameters are $r = 20\%, c = 15\%$, $\sigma_H = 0.15, b = 0.08$,			
$\rho = -0.20$, $\theta^* = 40\%$, $\lambda = 20\%$, $\kappa = 0.3\%$, $\mathbf{L}_R = 0.20$.			

The following table presents the comparison between two calibration, namely, with $\rho = 0$, and with $\rho = -0.20$.

TABLE 5.			
Comparison of Insurance Premiums			
LTV	$\rho = -0.20$	$\rho = 0$	Difference
0.80	0.00041	0.00145	0.00104
0.85	0.0045	0.0099	0.0044
0.90	0.0330	0.0579	0.0249
0.95	0.18139	0.2487	0.0674

The fourth column (difference) gives the difference between two calibrations and it can be seen that maxium difference (occurs when LTV is 0.95) is 0.06% which is not very significant.

6 Conclusion

In this paper, we price mortgage insurance premium using bivariate Binomial option pricing technique proposed by Hilliard et al (96). This technique allows us to use two state variables for mortgages, namely, interest rates and house prices. We first transform the two stochastic processes to processes with constant variance, then another transformation is made to make once-transformed processes uncorrelated. Thus after two transformations we obtain uncorrelated and constant-variance processes. So that we can safely use a three-dimensional binomial lattice to backward-price an option. Then we try to model the insurance contract and the insurance premium as a difference of the prices of two options. Using the lattice we model before we try to price the insurance premium.

The result we obtain with different LTV's and time to maturity is given in Table 2. In Table 3 we state the resu we compare our results of previous works and make comparison.The comparison shows that the difference between two studies is not higher than %0.06.

As it is summarized above we use fix-rate payment in our model. Similar study can be done to answer how mortgage insurance premium behave in a model with adjustable-rate mortgages (ARM)[8] which should be higher than the former as these mortgages can potentially impose larger payment burdens[9].

[8]Adjustable-Rate Mortgage (ARM) calls for resetting the interest rate periodically, in accordance with some appropriately chosen index reflecting short-term market rates.(Fabozzi F.J., Modigliani F. (1992))

[9]Dickie(1994) state that " in recent years ninety day delinquency for adjustable rate mortgages purchased by Fannie Mae has been roughly 50% to 100% higher than the delinquency rate on fixed rate mortgages."

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A Appendix A

B Appendix B

Consider the last term on the right hand side of (17):

$$
\begin{aligned}
(dZ_H + dZ_r)^2 &= d^2 Z_H + 2dZ_H dZ_r + d^2 Z_H \\
&= dt + 2\rho dt + dt \\
&= 2(1+\rho)dt
\end{aligned}
$$

and therefore

$$
\begin{aligned}
(dZ_H + dZ_r) &= \sqrt{2(1+\rho)dt} \\
&= \sqrt{2(1+\rho)}dz_1
\end{aligned}
$$

Similarly (18):

$$
(dZ_H - dZ_r) = \sqrt{2(1-\rho)}dt = \sqrt{2(1-\rho)}dz_2
$$

where $dZ_H dZ_r = \rho dt$.

Rewriting (17) and (18) gives:

$$
\begin{aligned}
dX_1 &= (b\mu + a)dt + b\sqrt{2(1+\rho)}dz_1 \equiv \mu_1 dt + \sigma_1 dz_1 \\
dX_2 &= (b\mu - a)dt - b\sqrt{2(1-\rho)}dz_2 \equiv \mu_2 dt - \sigma_2 dz_2
\end{aligned}
$$

where

$$
\mu_1 = (b\mu + a), \sigma_1 = b\sqrt{2(1+\rho)}
$$

and

$$\mu_2 = (b\mu - a), \sigma_1 = b\sqrt{2(1-\rho)}$$

The expression for μ_1 can be expanded as:

$$\mu_1 = (b\mu + a) = b\left[\frac{m^*}{\sigma_H} - \frac{S\sigma'_H}{\sigma_H^2}\right] + a$$

substituting m^*, and m yields:

$$\begin{aligned} &= b\left[\frac{[m - \sigma_H^2/2]}{\sigma_H} - \frac{S\sigma'_H}{\sigma_H^2}\right] + a \\ &= b\left[\frac{r \; - \sigma_H^2/2]}{\sigma_H} - \frac{S\sigma'_H}{\sigma_H^2}\right] + a \end{aligned}$$

Note that $r = R = \frac{X_1 - X_2}{2}$, $S = \frac{X_1 + X_2}{2b}\sigma_H$, $a = [\kappa(\theta - r(t)) - \lambda b] = [\kappa(\theta - \frac{X_1 - X_2}{2}) - \lambda b]$, and $\sigma'_H = 0$ we get the desired result

$$\mu_1 = b\left[\frac{[\frac{X_1 - X_2}{2}] - \frac{\sigma_H^2}{2}}{\sigma_H}\right] + \left[\kappa(\theta - \frac{X_1 - X_2}{2}) - \lambda b\right]$$

Similarly,

$$\mu_2 = b\left[\frac{[\frac{X_1 - X_2}{2}] - \frac{\sigma_H^2}{2}}{\sigma_H}\right] - \left[\kappa(\theta - \frac{X_1 - X_2}{2}) - \lambda b\right]$$