Quadratic Variation When Quoted Prices Change One Tick at a Time

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Abstract

For financial assets whose best quoted prices are almost always revised by the minimum price tick, this paper proposes an estimator of quadratic variation which is robust to microstructure effects. It compares the number of *alternations*, where quotes are revised back to their previous price, to the number of other jumps. Under an assumption of "uncorrelated alternation" the proposed statistic is then consistent as the intensity of jumps increases, and the price tick declines, indefinitely. The statistic is implemented across a range of applicable markets, which is enlarged by suitably rounding prices. A multivariate extension and a feasible central limit theory are developed.

JEL classification: C10; C22; C80

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1 Introduction

There is widespread evidence of persistence in financial assets' volatility. Therefore, estimating recent volatility furthers the desirable goal of forecasting volatility in the near future. In practice it is convenient to estimate the closely related quadratic variation (QV) in a realized price process, rather than its volatility directly. Andersen, Bollerslev, and Diebold (2005) and Barndorff-Nielsen and Shephard (2005b) provide surveys of the literature on this topic. The availability of rich second-by-second price data has encouraged high-frequency sampling when estimating QV. However, consistent estimation is significantly complicated at the highest frequencies by market microstructure effects. This paper points out features in many markets' microstructure which, when tested for positively, can be used as structural restrictions to control for this interference. This then leads to a new estimator for QV.

Many liquid financial markets "trade on a penny": i.e. their bid-ask spread is bid down to its regulatory minimum, the price tick (a penny, half a penny, etc.), almost all the time.[1] Empirically on such markets, the best bid and ask change through sporadic jumps by one price tick: so, they are pure jump processes of constant jump magnitude. Furthermore, the direction of their jumps often has a non-correlation property in its reversal pattern, termed in this paper "uncorrelated alternation". These easily testable features are, for example, found on the Chicago Board of Trade's (CBOT's) electronic market for 10-Year US Treasury Bond Futures, and the London Stock Exchange's (LSE's) market in Vodafone, which was its most active equity on a number of measures in 2004.

When these testable features are present, the paper proposes estimating QV from the best bid or the best ask with the statistic

$$nk^2 \frac{c}{a}, \tag{1}$$

where $n \in \mathbb{N}$ is the number of jumps in the best bid or ask, the constant $k > 0$ is the price tick, and $a \leq n$ is the number of *alternations*, i.e. jumps whose direction is a reversal of the last jump. Engle and Russell (2005) studies these. Jumps which do not alternate are *continuations*, and number $c = (n - a)$. Under some further technical assumptions (which do not rule out leverage effects) the statistic in (1) is consistent for

[1] On most exchanges, offers to buy or sell a financial asset must be priced at an integer multiple of the price tick, a quantity determined for each traded asset by the exchange.

the price's underlying QV. The term nk^2 is the QV of the observed price. This is an inconsistent, and normally an upwardly biased, estimate of underlying QV because of microstructure effects. However the upwards bias implies an excess of alternation, and in fact multiplying by the fraction c/a compensates consistently.

Consistency here is under a double asymptotic limit theory reflecting both the high-frequency and the small-scale of the market microstructure: in it the intensity of jumping grows without limit, and the squared magnitude of each jump diminishes at the same rate (see Delattre and Jacod (1997) for a related limit theory). This differs from the limit theory of Zhang, Mykland, and Aït-Sahalia (2005), Aït-Sahalia, Mykland, and Zhang (2005), Zhang (2004), Bandi and Russell (2004), Hansen and Lunde (2006) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004), which present consistent estimators of QV even if market microstructure effects are not small.

The necessary testable features do not apply on all markets, and exclude notably markets which have significant numbers of "double jumps", i.e. jumps of twice the price tick, in their quotes. For these markets, which typically do not trade on a penny, two rounding techniques are implemented in a later section, removing double jumps when applied to the bid, ask or mid-quote. The statistic is then robust to a range of microstructure effects including the bid-ask bounce, informed trading (see Kyle (1985), Glosten and Milgrom (1985) and Hasbrouck (1991)), volume-related effects (see Engle and Lange (2001)), and resiliency dynamics (see Biais, Hillion, and Spatt (1995), Coppejans, Domowitz, and Madhavan (2003) and Degryse, de Jong, van Ravenswaaij, and Wuyts (2003)).

The paper proceeds as follows: Section 2 presents the main theorem of the paper, as well as the relevant asymptotic limit theory. Section 3 then outlines the theorem's proof (detailed derivations are left to the Appendix). Section 4 applies the method to Vodafone, GlaxoSmithKline and Shell equity data on the LSE, and treasury bond futures at CBOT. Section 5 discusses corollaries and extensions, including feasible asymptotic limit theory and a related Monte Carlo study, while Section 6 concludes.

2 The model and main result

This section first prepares the ground for the main result, given in Section 2.4. The probability space $\{\Omega, \mathcal{F}, P\}$ is generated by three processes on $\mathbb{R}^+$: W, a standard Brownian motion, V, a pure jump process, and volatility σ, a Brownian semi-martingale with jumps. All jumps are finite activity. The focus of the paper will be on X, an underlying price, and Y, an observed price (e.g. bid or ask) defined thus:

$$(X, Y) = (W, V)_{[X]}, \tag{2}$$

where

$$[X]_t = \int_0^t \sigma_u^2 du. \tag{3}$$

So W and V are subordinated by the process $[X]$, and X is a time-changed Brownian motion with stochastic volatility σ,[2] which may have arbitrary serial dependence. X may have leverage effects, i.e. W and σ may be dependent. Processes such as W and V which are subordinated by the time-change $[X]$ will be said to evolve in "business time", while X and Y evolve in "calendar time"(see Oomen (2004) for more on this terminology).

For some $T > 0$, only $\{Y_t : 0 \leq t \leq T\}$ is observed. Y has a random initial value. The quantity to be estimated is the QV of X over the period that Y is observed, namely $[X]_T$. As X is not observed, nor is $[X]_T$. Y deviates from X by a microstructure effect, the process ϵ, which we define in calendar time by

$$\epsilon = Y - X. \tag{4}$$

Hence, $\epsilon = (V - W)_{[X]}$, and so $(V - W)$ is the microstructure effect viewed in business time. The following two conditions recur throughout the paper.

Definition The microstructure is stationary in business time if $(V - W)$ *is stationary.*

Definition The microstructure has no leverage effects if

$$V|W \perp\!\!\!\perp \sigma|W, \tag{5}$$

where $\perp\!\!\!\perp$ indicates independence.

While allowing leverage effects in X, this means that in business time the microstructure effect is conditionally independent of current volatility. So, the frequency, not the magnitude, of quote changes grows with increased volatility: which is reasonable where markets trade on a penny or thereabouts.[3]

[2] For more on stochastic volatility, see, for example, the reviews in Ghysels, Harvey, and Renault (1996) and Shephard (2005, Ch 1).

[3] An alternative definition for there to be no leverage effects in the market microstructure, not adopted

2.1 Constant observed jump magnitude

The observed pure jump process, Y, may be specified by

$$Y_t = Y_0 + \int_0^t G_u dN_u, \tag{7}$$

where N is a simple[4] counting process and G is an adapted process that only takes values $\pm k$ for some $k > 0$. The QV of Y is

$$[Y]_t = \int_0^t G_u^2 dN_u = k^2 N_t, \tag{8}$$

a stochastic process. Decompose the process N by

$$N = A + C, \tag{9}$$

where A and C are counting processes. The *alternation* process, A, counts the jumps in Y which have opposite sign to the one before, and the *continuation* process C counts jumps that continue in the same direction as the one before. Both are adapted to Y. Notice that as N is simple, arrivals of A and C at the same time are not possible, so the decomposition is unique. Exclude the first jump in Y. For all $i \in \mathbb{N}$ let t_i be the time of the i'th jump in Y. Define the random sequence $Q = \{dA_{t_i} - dC_{t_i} : i \in \mathbb{N}\}$. So Q records $+1$ for an alternation and -1 for a continuation.

Definition Y has Uncorrelated Alternation if Q has zero first-order autocorrelation.

2.2 Technical assumptions

Identification Assumption Given two events observable before any jumping time t_i, $H_1 \in \mathcal{F}_{t_i-}$ and $H_2 \subset H_1$,

$$\{\ E(Y_{t_i}|H_1) = E(Y_{t_i}|H_2)\ \} \leftrightarrow \{\ E(X_{t_i}|H_1) = E(X_{t_i}|H_2)\ \}. \tag{10}$$

Thus, if H_2 adds (no) new information to H_1 concerning the likely direction of Y's next jump, it adds something (nothing) new about the level of X.

here, would be one in "calendar time" :

$$\left\{ \int_0^t \frac{\epsilon_u}{\sigma_u} du \right\}_{t=0}^{\infty} \mid W \quad \perp\!\!\!\perp \quad \sigma \mid W. \tag{6}$$

This would imply that the normalized increments in ϵ were independent of current volatility, meaning that, for example, the mean magnitude of revisions in quoted prices would grow with increased volatility.

[4] I.e. the probability of observing two or more events in a small period of time, when divided by the probability of observing one event, is second order.

Buy-sell Symmetry Let $\{w, v\}_s$ be a realization of (W, V) up to business time s. The microstructure is buy-sell symmetric if

$$dV_s|\{w, v\}_s \quad \sim \quad -dV_s|\{-w, -v\}_s. \tag{11}$$

2.3 Asymptotic limit theory

A long sample invites the time series econometrician to suppose that the sample were of "infinite" length. This is behind much sampling theory used in macroeconomics and financial economics. Of course, in practice the data is finite and so these asymptotics provide an approximation, whose accuracy can be assessed through the simulation of realistic cases, or higher order asymptotic expansions.

Similarly, high frequency market microstructure data invites the asymptotic thought experiment that, given an underlying price process, the microstructure had evolved "infinitely" fast, with "infinitely" small jumps. Delattre and Jacod (1997) exemplify such an approach. This implies a double asymptotic theory where an unbounded increase in jumping intensity and decline in jumping magnitude have some relative rate. The rate is here proposed so that $[Y]_T$ can, in line with observed data, have a non-zero limit in probability. The next part formalizes this intuitive idea in terms of a scaling constant, $\alpha \in \mathbb{R}^+$, which converges to zero from above.

2.3.1 Formal asymptotic theory

Assume that the microstructure is stationary and has no leverage effects. So the probability measure admits the following factorization:

$$P(W, V, \sigma) = P(V|W) \; \times \; P(\sigma|W) \; \times \; P(W). \tag{12}$$

Define the process W^α by

$$W_t^\alpha := \frac{1}{\alpha} W_{\alpha^2 t}, \tag{13}$$

So, for $\alpha < 1$, the functional $W \rightarrow W^\alpha$ slows but normalizes W so that W^α is also standard Brownian motion. Define a new conditional probability measure $P_\alpha(V|W)$ by:

$$P_\alpha(V|W) := P(V^{1/\alpha}|W^\alpha). \tag{14}$$

The asymptotic theory studies

$$\lim_{\alpha \to 0} \left\{ P_\alpha(V|W) \; \times \; P(\sigma|W) \; \times \; P(W) \right\}, \tag{15}$$

as an approximation to (12).

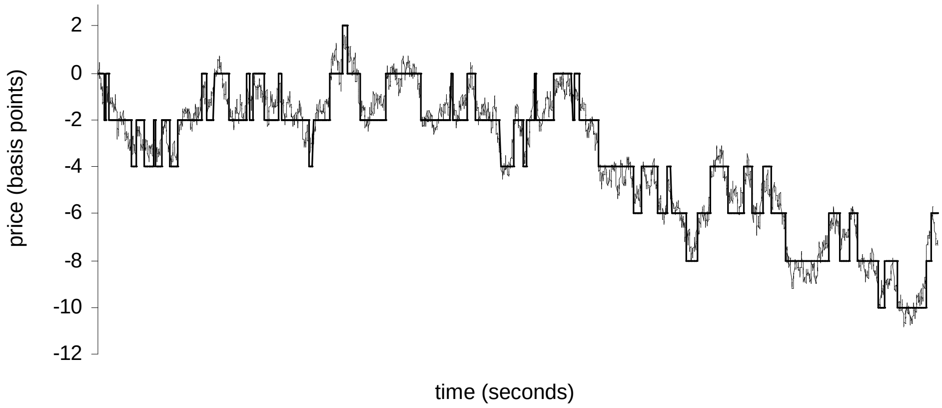

Figure 1: A simulation of this paper's proposed model. It shows an asset's observed price, which jumps, and its continuous underlying price, here a scaled Brownian motion. The observed price has a propensity to alternate.

Intuition Suppose that $\alpha < 1$. The conditional measure P_α can be understood as the result of a three stage process: first W is slowed and scaled up. Second, V evolves stochastically according to the model, conditional on the slowed and expanded W. Third, W and V are speeded back up and scaled back down.

Therefore the measure $P_\alpha(V|W)$ makes more likely, for given X, realizations of Y with more jumps, of smaller magnitude. Indeed, with probability 1, $N_T \to \infty$ as $\alpha \to 0$. Note that this asymptotic formalization is also applicable in cases when Y is not simply a pure jump process. This manipulation preserves the stationarity of ϵ in business time.

2.4 The main result

Theorem 2.1 *Suppose that*

(A) Y has Uncorrelated Alternation,

(B) Y always jumps by a constant $\pm k$,

(C) ϵ has no leverage effects, is stationary in business time, is ergodic, and $E(\epsilon) = 0$.

(D) Y always jumps towards X, and

(E) The Identification Assumption and Buy-Sell Symmetry hold.

Condition on $[X]_T$, so that T is random. Then

$$[Y]_T \frac{C_T}{A_T} \tag{16}$$

is a consistent estimator of $[X]_T$ *as* $\alpha \to 0$*. The limiting distribution as* $\alpha \to 0$ *of*

$$\sqrt{N_T}\left[\frac{[Y]_T \frac{C_T}{A_T}}{[X]_T} - 1\right] \tag{17}$$

exists and is normal. Its variance depends on the market's short-run order dynamics, and is detailed later in Proposition 3.5.

Proof. Section 3 provides the proof of this Theorem through a series of propositions, whose detailed derivations are left to the appendices. ■

Figure 1 shows a process satisfying the Theorem's assumptions.

Discussion of the result The result is semi-parametric because it does not refer to the dynamics or the intensity of N. The proposed estimator is easy to calculate. It multiplies together two components: $[Y]_T$, which is equal to $N_T k^2$, and the ratio C_T/A_T (which may be more or less than 1). Many jumps are indicative of high volatility unless most of them are alternations, a possibility which the observed proportion of alternations to continuations provides a means to account for. Since it has no fixed observation frequency, the statistic does not encounter systematic biases due to intraday seasonality.

Discussion of the assumptions Assumptions (A) and (B) can be tested empirically. (A) states that the likelihood that a jump is an alternation does not depend on whether the last jump was. It may be tested via a regression of Q on itself lagged. (B), which assumes a constant jump magnitude, is true of many markets' quotes, including two of the markets studied in the empirical section, Section 4. Section 4 goes on to treat cases where (B) and (A) do not hold directly using various rounding techniques.

The assumptions (C), (D) and (E) cannot be tested. (C) states that viewed in "business time" the microstructure effect is ergodic and independent of current volatility. Thus, while at times it evolves fast, these are exactly the times when X also evolves fast. It does not preclude leverage effects in X. (D) rules out transitory increases to $|\epsilon|$ through noise or other trading. Finally, (E) is innocuous.

Relationship to the existing literature The availability of rich second-by-second data has encouraged high-frequency sampling of prices when measuring their QV. The benchmark case is to compute the observed price's Realized Variance (RV) at some high frequency. This (in calendar time) is calculated by breaking up a period of time, e.g. a trading day, into many intervals of equal length, then squaring the observed returns over these intervals, and adding them up. Barndorff-Nielsen and Shephard (2002) provides an asymptotic limit theory for RV as it approaches QV with faster sampling but before market microstructure becomes a central concern (see also Jacod (1994) and Jacod and Protter (1998)). In the current framework, at the highest frequency RV is $[Y]_T$.

However, researchers have found that a price's RV at high frequencies typically deviates significantly from its RV at low frequencies, see Zhou (1996), Andreou and Ghysels (2002) and Oomen (2002). This has been attributed to microstructure noise, i.e. ϵ, in Aït-Sahalia, Mykland, and Zhang (2005), Zhang, Mykland, and Aït-Sahalia (2005), Zhang (2004), Bandi and Russell (2004), Hansen and Lunde (2006) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004). Like these, the current work makes a correction to RV at high frequency so as to account for microstructure noise.

The idea that observed prices are pure jump processes, which deviate from a fundamental price, is already present in Oomen (2004) and Zeng (2003). This perspective explains two interrelated puzzles. First, if prices are pure jump processes, then the observed asymptotic behavior of bipower variation, the influential statistic introduced in Barndorff-Nielsen and Shephard (2006), which converge to zero with finer sampling, is explicable. Second, when studying quotes data, Hansen and Lunde (2006) find that at high frequencies RV is at times a downwards-biased estimator of QV. They show that this result implies a negative covariation between efficient returns and the error due to microstructure effects. They also document time-dependence in the error. A pure jump process can account for these features: the local infrequency of jumps implies both serial correlation in the error and instantaneously negative covariation between the error and efficient price.

The use of a time-change argument in the proof of the Theorem relates to Barndorff-Nielsen and Shephard (2005a).

3 Proof of Theorem 2.1

Definition Conditional on given $[X]_T$, let R be the ratio

$$R = \frac{[X]_T}{E\,[Y]_T}. \tag{18}$$

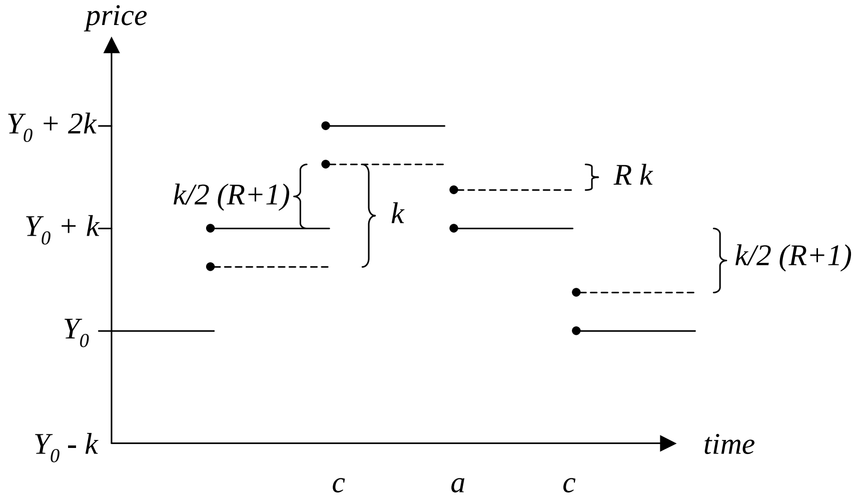

Figure 2: The solid line shows Y, while the dashed line shows Z. The letters on the time axis indicate if the jump is an alternation or a continuation. The diagram is of an example illustrating the relative contribution to the QV of Z by alternations and continuations.

Note that $E\ [Y]_T$ exists because N is simple, and that T is a random time. Under Assumption (C), R is invariant to the conditioning information $[X]_T$.

Proposition 3.1 *Suppose that Assumptions (B),(C) and (D) of Theorem 2.1 hold. The error just before the i'th jump is ϵ_{t_i-}. Taking the ergodic expectation, for all i*

$$E[\mid \epsilon_{t_i-} \mid] = \frac{k}{2}\left[R+1\right]. \tag{19}$$

Proof. See Appendix A. ■

Discussion This proposition implies that the expected magnitude of the error, measured just before a jump, is an increasing affine function of $R = \frac{[X]_T}{E\ [Y]_T}$. Its intercept is $\frac{k}{2}$, so that if R is very near zero, e.g. if X is almost constant, then Y simply jumps between $X \pm \frac{k}{2}$. As $[X]_T$ increases, the expected error magnitude increases.

Proposition 3.1 provides a unbiased estimate of $\mid \epsilon_{t_i-} \mid$ while under Assumption (D) the direction of the jump at t_i gives the sign of ϵ_{t_i-}. Combining these, an unbiased estimate of ϵ_{t_i-} itself is available. Equally, X_{t_i} can be estimated without bias by adding or subtracting $E[\mid \epsilon_{t_i-} \mid]$ to/from Y_{t_i-}, depending on the direction Y jumps at t_i. The next definition gives the name Z to this conditional estimation process, which is illustrated in Figure 2.

Definition For each of Y's jumping times, t_i, define Z_{t_i} by

$$Z_{t_i} = E[X_{t_i} \mid Y_{t_i}\ , G_{t_i}\ , R\ ]. \tag{20}$$

(Recall that $G_{t_i} = \pm k$ is the jump in Y at t_i.) Extend the sequence $\{Z_{t_i} : i \in \mathbb{N}\}$ rightwards to a càdlàg pure jump process Z. Note that Z is not observed because R is not observed. The evolution of Z is described in Figure 2, which also illustrates the following lemma.

Lemma 3.2 *The Quadratic Variation process for Z, denoted $[Z]$, is a linear combination of the processes A and C given by*

$$[Z] = k^2(C + R^2 A). \tag{21}$$

Proof. When Y jumps by continuing in the same direction as the last jump, Z jumps by k. When Y jumps by alternating in direction, Z jumps by Rk. This follows from simple calculation, and is easily seen in Figure 2. The QV of Z is the sum of its squared jumps. The lemma now follows. ■

Definition A process S has Ideal Error if conditional on any $[X]_T$,

$$E\ [S]_T\ = [X]_T, \tag{22}$$

where the expectation is ergodic.

Proposition 3.3 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1, as well as the Identification Assumption, hold. Uncorrelated Alternation then implies that Z has Ideal Error.*

Proof. See Appendix B. ■

Uncorrelated Alternation may be tested simply by regressing Q linearly on itself lagged, and testing that the regressor is significant.

Proposition 3.4 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then, conditional on $[X]_T$,*

$$E[A_T R - C_T] = 0, \tag{23}$$

and R has the Method of Moments estimator

$$\hat{R} = \frac{C_T}{A_T}. \tag{24}$$

(Define $\hat{R} = 0$ if $C_T = A_T = 0$).

Proof. See Appendix C. The main case is when Y does not have Ideal Error. In a sketch, as Z has Ideal Error, and $[X]_T = R\ E[k^2 N_T]$,

$$E[k^2(C_T + A_T R^2)] = RE[k^2 N_T]. \tag{25}$$

Thus the expectation of a quadratic in R is 0:

$$E[A_T R^2 - N_T R + C_T] = 0. \tag{26}$$

The quadratic has roots at 1 and C_T/A_T. But $R \neq 1$ since Y does not have Ideal Error. Hence (23) is true. ■

So, recalling that $[X]_T = R\ E[Y]_T$, the proposed estimator of $[X]_T$ is

$$\hat{R}[Y]_T. \tag{27}$$

Denoting by $\hat{Z}$ the estimate of the process Z constructed by replacing R with $\hat{R}$ in (20), straightforward algebra shows that

$$\hat{R}[Y]_T = [\hat{Z}]_T. \tag{28}$$

The final proposition in this section provides the asymptotic limit theory for this estimator, proving its consistency.

Proposition 3.5 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then conditionally on $[X]_T$, the following limit theory applies:*

$$\lim_{\alpha \to 0} \sqrt{N_T} \left(\frac{\hat{R}[Y]_T}{[X]_T} - 1 \right) \sim N(0, UMU'), \tag{29}$$

where U is the pair $(1, \frac{(1+R)^2}{R})$ and M is the long-run variance matrix of the stationary time series of pairs, Π:

$$\Pi = \left\{ \left(\frac{[X]_{t_i} - [X]_{t_{(i-1)}}}{E([X]_{t_i} - [X]_{t_{(i-1)}})}\ ,\ \frac{Q_i + 1}{2} \right)\ : i \in \mathbb{N} \right\}. \tag{30}$$

The left hand term here is the elapsed QV in X between the $(i-1)$th and ith jumps in Y, once de-averaged. As previously defined, Q_i takes value $+1$ if the ith jump in Y is an alternation, and -1 if it is a continuation.

Proof. See Appendix D. ■

This asymptotic limit theory is infeasible because the elapsed QV is not directly observed, ruling out estimates of Π and so of M.

Proof of Theorem 2.1 Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. (Then Z may be constructed as in Figure 2.) If in addition Assumptions (A) and (E) hold, then Proposition 3.3 shows that Z has Ideal Error. Therefore Propositions 3.4 and 3.5 apply and the Theorem follows.

4 Empirical implementation

4.1 Vodafone

This part implements the proposed estimator for Vodafone stock traded on the LSE's electronic limit order book, called SETS. Vodafone was the LSE's most heavily traded stock (by dollar value) in 2004. We study it over the period of seven months from August 2004 until the end of February 2005. This period comprised 147 trading days running from 8:00am to 4:30pm.[5] Quotes were timed to the nearest second. Vodafone's best bid was revised 17,060 times over the sampled period, while its offer was revised 17,167 times, an average of 116 times per day. The relative infrequency of changes in Vodafone's quoted prices make the task of estimating QV particularly challenging, and highlights the importance of using the data at the highest manageable frequency.

4.1.1 Specification Testing

In line with, for example, Engle (2000), the first 15 minutes of the trading day were excluded. The remaining price series for both the bid and the ask were tested for uncorrelated alternation in a first order autoregression. Days were broken at 12pm into the morning and the afternoon, producing 294 periods. For the best ask 6.8 per cent of periods failed the LR test at 5 per cent. For the best bid the figure was 8.5 per cent. While ideally these numbers would be close to 5 per cent, in reality a small minority of days experience abnormal market microstructure due to substantial price jumps, news announcements, and other information effects. Finally, only 0.5 per cent of jumps in the

[5]Except for 24 December and 31 December, when markets closed at 12:30pm.

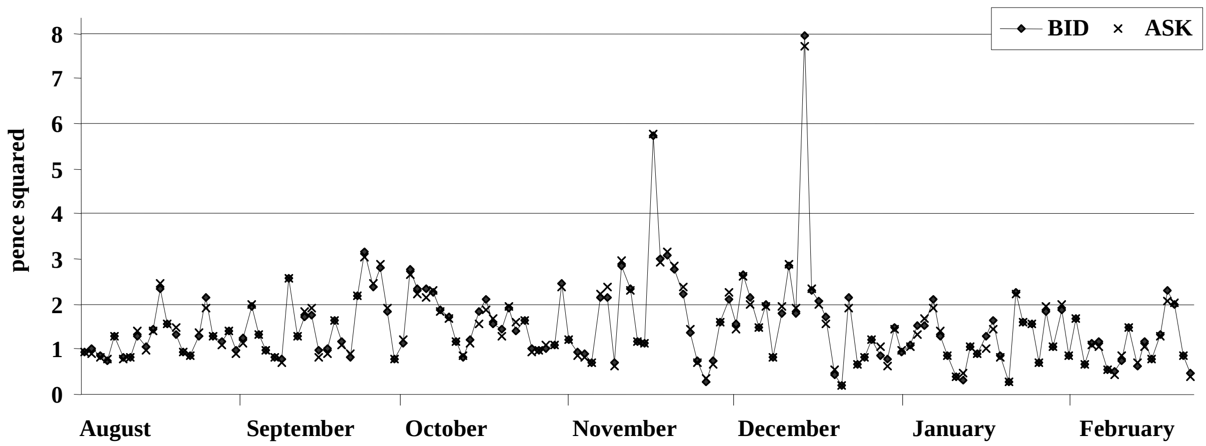

Figure 3: The estimated QV for Vodafone on each trading day from August 2004 to February 2005. Isolated crosses show the estimated QV of the best ask, while diamonds joined by lines show the same for the best bid.

best bid or ask were double (or more-than-double) jumps. In conclusion, only during infrequent brief interludes was the model found to be misspecified.

4.1.2 Results

The time series of daily estimated QV is shown in Figure 3. The time series both for the bid and for the ask are plotted: and the two series track one another closely. Andersen, Bollerslev, Diebold, and Labys (2000) introduced a graphical technique known as the volatility signature plot. This plots the Realized Variance of a price process as a function of the frequency at which it is sampled. Legibility is enhanced by plotting sampling frequency on a log-scale. Finally, by analyzing a long enough time series, it is hoped that (even at its right-hand end) the shape of the schedule will be reasonably stable due to large number effects. Figure 4 shows for the current data, volatility signature plots of Y, and the process $\hat{Z}$, which approximates Z and is constructed using $\hat{R} = C_T/A_T$. Six days were excluded from the figure, since they contained large jumps in the price. These were Christmas Eve 2004, New Year's Eve 2004, and the third Fridays in November 2004, December 2004, January 2005 and February 2005.

Under the assumptions of Proposition 3.3, passing a test for uncorrelated alternation

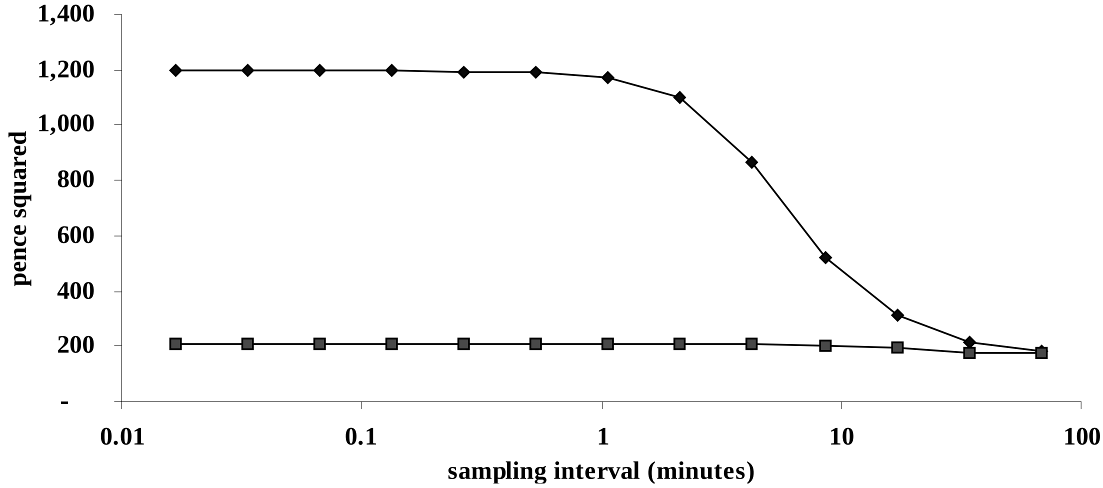

Figure 4: Volatility signature plots for Vodafone's best bid price from August 2004 to February 2005. The diamonds show the estimated RV of the quote at various observation frequencies. The squares show the estimated RV of the transformed quote, $\hat{Z}$, at various frequencies.

implies that Z has Ideal Error. Therefore loosely, this test can be interpreted as sufficient for the hypothesis that the volatility signature plot of Z is flat. Inspection of Figure 4 suggests this may at least be so here of $\hat{Z}$.

4.2 CBOT Treasury Bond Future

We turn to another asset that trades on a penny. The data contains the prices of the 10-Year Treasury Bond Future at the Chicago Board of Trade (CBOT) on 29 July, 30 July and 2 August 2004 (the dates span a weekend), as displayed in Figure 6. Open trading on the electronic limit order book runs from 7am to 4pm, and there are often macroeconomic announcements around 7:30am. We therefore study the market from 7:32am to 4pm on each day. The best ask was studied, which is quoted in quantities representing $\frac{1}{12{,}800}$ of the contract's nominal value, \$100,000. The price level on this market was approximately 14,000 at this time. The price tick was two. The best ask was revised 3,254 times over the sampled interval, or on average 1,047 times per day. Of these, 20 jumps, or 0.6 per cent, were greater than the price tick.

Each day was divided into the period before 9:00am, and the period after. Small intervals containing jumps greater than the price tick were excised, principally a 25-minute period around 9:00am on 30 July. At this time there was a sudden period of

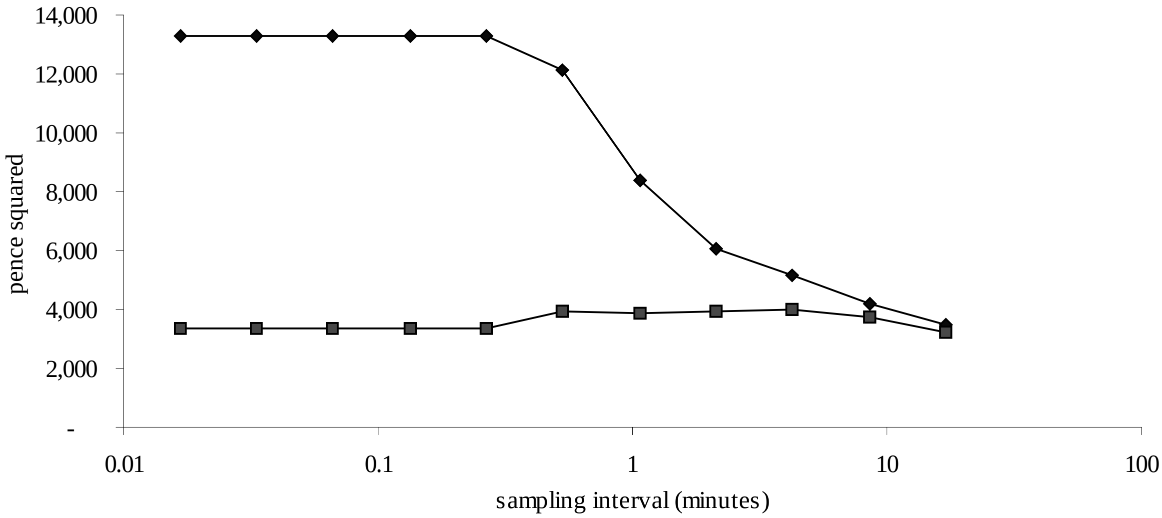

Figure 5: Volatility signature plots for the best ask on the CBOT limit order book for the 10-Year Treasury Bond Future. The diamonds show the estimated RV of the quote at various observation frequencies. The squares show the estimated RV of the transformed quote, $\hat{Z}$, at various frequencies.

very high volatility, perhaps due to a public announcement (See Figure 6). For three of the six periods the null hypothesis of *no first order autocorrelation* was accepted at a confidence level of 5 per cent. For the other three periods, it was accepted at 2 per cent. The data's volatility signature plot, as well as the one for $\hat{Z}$, are presented in Figure 5. The latter's flatness provides further corroboration of the hypothesis that Z has Ideal Error. QV estimation results for CBOT are reported in Table 3.

4.3 GlaxoSmithKline

Vodafone is one of the only equities on LSE SETS which trades on a penny. Where assets do not trade on a penny, the model is typically mis-specified. GlaxoSmithKline (GSK) provides an example of this: over the 147 days from August 2004 to February 2005, the average bid-ask spread was 1.15 pence, but the price tick was 1 pence. Double jumps are correspondingly more prevalent than for Vodafone: 4.9 per cent of changes in the best bid were double jumps, as were 4.6 per cent of changes in the best ask. As it stands, the model is therefore badly specified.

To make the data applicable, a initial preparation step is required. Two techniques are proposed here. First, the quote may be rounded down (or up) to the nearest even

	Per cent of half-days failing spec. test at 5 per cent	Jumps per day	Cont. / Alt.	Estimated daily QV (pence^2)
Bid				
rounded to the nearest 2 p	3.7	210	0.19	156
sluggishly rounded to 2 p	7.8	61	0.62	151
Ask				
rounded to the nearest 2 p	6.1	211	0.19	158
sluggishly rounded to 2 p	6.5	60	0.65	155
Mid-quote				
rounded to the nearest 1.5p	7.8	278	0.25	156
sluggishly rounded to 1.5p	8.2	77	0.89	155

Table 1: Results of specification testing and estimation for GSK quotes, rounded.

(or odd) multiple of the price tick. Second, the quote may be "sluggishly rounded": call the observed data Y and the prepared data $\tilde{Y}$. Obtain $\tilde{Y}$ from Y by setting $\tilde{Y}_0 = Y_0$, and letting $\tilde{Y}$ jump an amount $2k$ towards Y whenever they differ by $2k$ or more. Both these techniques result in processes that almost only contain double jumps, and which are therefore amenable to the present model. Moreover, the techniques can be applied to mid-quote, whose minimum price increment is half the price tick. Finally, a larger multiple of the minimum price increment than 2 can also be used. GSK's quoted price changes were more frequent than for Vodafone: the best bid was revised 58,787 times over period, while its ask was revised 59,093 times, an average of 401 times per day. Its mid-quote changed on average 588 times per day. Therefore, these rounding techniques can still be expected to produce reasonably rich data sets, at least in comparison to Vodafone.

4.3.1 Specification testing and results

For each day in the studied period, the bid, ask and mid-quote were separately prepared using both rounding and sluggish rounding. The results of specification testing and estimation are presented in Table 1. With the same provisos as for the Vodafone data, all the methods of preparation produce fairly well specified models. As documented in Table 1, although the six preparations techniques result in differing numbers of jumps per day, and substantially differing propensities to alternate, they imply very similar

	Per cent of half-days failing spec. test at 5 per cent	Jumps per day	Cont. / Alt.	Estimated daily QV (pence^2)
Bid				
rounded to the nearest 0.5 p	6.1	290	0.18	13.3
sluggishly rounded to 0.5 p	8.5	85	0.63	13.3
Ask				
rounded to the nearest 0.5 p	4.8	281	0.19	13.6
sluggishly rounded to 0.5 p	4.8	86	0.62	13.2
Mid-quote				
rounded to the nearest 0.375 p	6.1	381	0.24	13.2
sluggishly rounded to 0.375 p	8.8	105	0.89	13.2

Table 2: Results of specification testing and estimation for Shell quotes, rounded. Shell's price tick is 0.25 pence.

estimates of underlying QV. In applications, it would be advisable to average all six estimates. Table 2 provides the same analysis for a third LSE share, Shell.

5 Extensions and corollaries

This section first discusses a filtering technique which relates closely to the Main Theorem. It then shows how when certain further restrictions can be imposed on a well-specified model, the central limit theory presented in Proposition 3.5 becomes feasible, permitting inference about the estimated QV. This is followed by an extension of the model to the bivariate case where two simultaneous price processes have correlated returns.

5.1 A filter

Under the assumptions of Theorem 2.1, first constructing $\hat{Z}$ provides a better estimate of X, the underlying price, than using Y directly. In this sense, $\hat{Z}$ thus constructed is a useful "filter" for Y.

The QV of $\hat{Z}$, $[\hat{Z}]_T$, differs from the proposed statistic, $k^2 N_T C_T / A_T$, if the data contains short episodes of model mis-specification with large quote revisions or swings due to e.g. public announcements, since it attenuates but does not disregard double jumps (the two statistics are identical if there is no mis-specification). In some circumstances

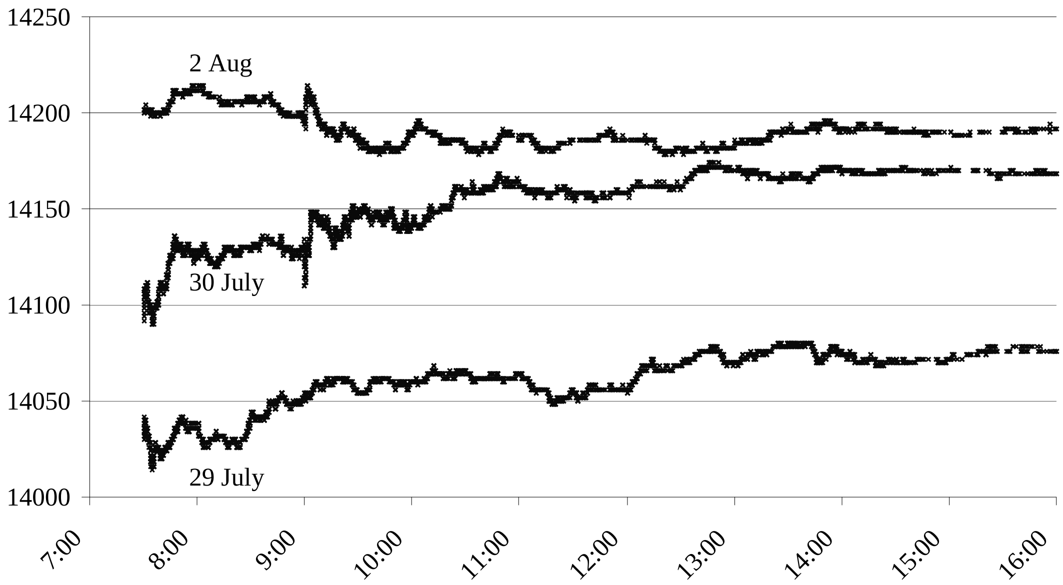

Figure 6: The price-path taken by the best bid for the 10-Year Treasury Bond Future on 29, 30 July and 2 August 2004.

$[\hat{Z}]_T$ may be a preferable statistic. The CBOT data provides an example of this. Its price path is shown in Figure 6: around 9am on 30 July and 2 August, the eye observes periods of heightened volatility including double jumps: $[\hat{Z}]_T$ includes a contribution to QV from the large jumps at that time, but $k^2 N_T C_T / A_T$ does not. Their discrepancy is reported in Table 3.

	C_T	N_T	$\hat{R}$	$k^2 N_T \frac{C_T}{A_T}$	$[\hat{Z}]_T$
29 July	176	847	0.26	889	889
30 July	316	1,592	0.25	1,577	1,714
2 August	140	815	0.21	676	778
Total	632	3,254	0.24	3,142	3,379

Table 3: Estimation results for the 10-Year Treasury Bond Future on 29, 30 July and 2 August 2004 (best ask). The discrepancy in some values of the two right-hand columns is due to the short episodes of mis-specification described in the text.

5.2 A feasible limit theory when volatility is constant

The asymptotic limit theory of Proposition 3.5 is infeasible because Π is not observed, and therefore its long-run variance matrix, M, cannot be estimated. However, circumstances may exist where the econometrician may reasonably suppose the process σ to be constant. Then the elapsed QV in X between jumps at t_i and t_{i-1} is given by

$$[X]_{t_i} - [X]_{t_{(i-1)}} = \sigma^2(t_i - t_{i-1}), \tag{31}$$

and, de-averaged,

$$\frac{[X]_{t_i} - [X]_{t_{(i-1)}}}{E([X]_{t_i} - [X]_{t_{(i-1)}})} = \frac{(t_i - t_{i-1})}{T} E(N_T). \tag{32}$$

Substituting N_T for $E(N_T)$ gives an estimate $\hat{\Pi}$, on which the Newey and West (1987) method, and other long-run variance estimation techniques, can be used to estimate M.

5.3 A feasible limit theory when Y follows Sluggish Rounding

Where the assumption of constant volatility is untenable, nevertheless inference may still be feasible by assuming a specific dynamic structure for the order flow. This approach is exemplified by Sluggish Rounding, which we now introduce.

When prices jump by a constant increment, it would seem reasonable to view them as resulting from rounding off a continuous process to the nearest penny, half-penny etc. This approach is taken in Hasbrouck (1998) and Hasbrouck (1999). It is also present in Zeng (2003) (where the underlying price is corrupted by noise, then rounded). An appropriate central limit theory is provided in this context by Delattre and Jacod (1997). In these papers, prices are discretely sampled. However, in the current setting of continuous sampling, rounded-off Itô processes must have QV either of zero, or of ∞, the latter with positive probability (provided prices are unbounded). This is because when an Itô process crosses a rounding threshold, with probability 1 it does so infinitely more times in the next instant. While retaining continuous sampling, this problem can be avoided by introducing a "sluggishness", whereby any threshold for rounding observed prices up by one increment exceeds by a small margin the threshold for rounding them back down.

Definition Y evolves according to Sluggish Rounding if there exists $\rho > \frac{k}{2}$ such that Y jumps towards X by amount k whenever $|X - Y| \geq \rho$.

For Y to have finite activity, it cannot be that

$$0 < \rho \leq \frac{k}{2}, \tag{33}$$

for then any single jump would precipitate an infinite flurry. In the case where $\rho = k$, Y jumps to exactly the value of X whenever X reaches $Y \pm k$.

Proposition 5.1 *Suppose that Y evolves according to Sluggish Rounding. Then it has Uncorrelated Alternation. Furthermore, Π is an i.i.d. sequence and*

$$UMU' = \frac{2}{3R}(1 + 4R + 2R^2). \tag{34}$$

Therefore UMU' may be estimated consistently by replacing R in (34) with $\hat{R} = C_T/A_T$.

Proof. See Appendix E. ■

A simulation of Sluggish Rounding 10,000 times based on $R = 0.25$, $k = 2$ and $[X]_T = 900$ – parameters which are in line with the CBOT data – is suggestive of good small sample properties. In particular, the truth was rejected at 1% on 1.2% of runs, at 5% on 4.8% of runs, and at 10% on 9.7% of runs (standard errors were estimated one run at a time using Proposition 5.1). Logarithms were taken at the lower tail to mitigate the influence of the lower bound at zero.

When calculated using Proposition 5.1, the quantity $\sqrt{\frac{UMU'}{N_T}}$ estimates the standard deviation of $\frac{[\hat{X}]_T}{[X]_T}$. Implemented daily for Vodafone from August 2004 to February 2005, it was on average 24 per cent. For Shell's rounded mid-quote and the CBOT quote data it averaged 13 and 8 per cent respectively.

5.4 Estimating bivariate covariation

It seems plausible that where the returns of two financial assets are positively correlated, the value of a portfolio containing both might have disproportionately many continuations. This intuition is supported by the following formalization. Let (X_1, Y_1) and (X_2, Y_2) be the models of two asset prices, satisfying the assumptions of Theorem 2.1, and let (X_1, X_2) be a bivariate Itô process. Note that without loss of generality they can be scaled so Y_1 and Y_2 have the same jump size, k. The quantity of interest is the covariation of X_1 and X_2, written

$$[X_1, X_2]_T = \text{p-lim} \sum_{j=1}^{M} [X_1(t_j) - X_1(t_{j-1})][X_2(t_j) - X_2(t_{j-1})], \tag{35}$$

where $\{0 = t_0, t_1, \ \ldots \ t_M = T\}$ is a lattice on $[0, T]$ whose mesh tends to zero in the limit, and p-lim denotes that limit in probability, as studied in Barndorff-Nielsen and Shephard (2004). Estimators of this quantity are proposed in Hayashi and Yoshida (2005) and Sheppard (2005).

Corollary 5.2 *Suppose that the models* $(X_1 + X_2, Y_1 + Y_2)$ *and* $(X_1 - X_2, Y_1 - Y_2)$ *each satisfy the Assumptions of Theorem 2.1 and, conditional on* (X_1, X_2)*,* Y_1 *and* Y_2 *are independent. Write* C^+ (A^+) *for the number of observed continuations (alternations) in* $Y_1 + Y_2$ *before time* T*. Define* C^- *and* A^- *analogously for* $Y_1 - Y_2$*. Then*

$$\frac{1}{4}\left(\frac{C^+}{A^+} - \frac{C^-}{A^-}\right)\left(\ [Y_1]_T + [Y_2]_T\ \right) \tag{36}$$

is a consistent estimator of $[X_1, X_2]_T$*.*

Proof. See Appendix G. ■

The corollary has the interpretation that, asymptotically, the covariation is positive if and only if there are more continuations in the summed price processes than in the differenced prices. However, the assumptions of this corollary are strong, and it is not expected that they would hold of all data.

6 Conclusion

This paper views the observed price as a pure jump process whose deviations from an underlying stochastic process are stationary in business time. Noting that on many markets the amount by which quotes jump is constant and equal to the price tick, it proposes a new estimator for the underlying price's QV which down-weights the quoted price's observed QV by a factor that takes into account its propensity to alternate. Provided that alternation is uncorrelated at the first order, the estimator is shown to be consistent in an appropriate asymptotic theory. Simple rounding techniques substantially widen the range of applicable price processes and markets. The estimator is implemented for some UK equities and a US future.

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A Proof of Proposition 3.1

The proposition states: Suppose that Assumptions (B),(C) and (D) of Theorem 2.1 hold. Condition on $[X]_T$, letting T be a random time. Let R be the ratio $\frac{[X]_T}{E\ [Y]_T}$. The error just before the ith jump is ϵ_{t_i-}. Taking the ergodic expectation, for all i

$$E[\mid \epsilon_{t_i-} \mid] = \frac{k}{2}\,[R+1]\,. \tag{37}$$

Proof Let $u = V - W$ be the microstructure effect in business time. The proof proceeds by equating the ergodic variances of u (equivalently, of ϵ) at (i.e. just after) two subsequent jumps, at random times, say the second and third, at times t_2 and t_3. For all t, $Var(u_t) = E(u_t^2)$. Define $\bar{\lambda}$ as the ergodic or average intensity of jumping in business time. Define $\{w_i\}$ and $\{v_i\}$ as the respective sets of random increments in W and V:

$$w_i = W_{t_i} - W_{t_{i-1}}, \tag{38}$$

$$v_i = V_{t_i} - V_{t_{i-1}}, \tag{39}$$

so that v_3 is equal to the jump at t_3, i.e. $\pm k$, and $\{w_i\}$ are i.i.d. of known variance:

$$w_i \sim N(\ 0\ , t_i - t_{i-1}). \tag{40}$$

It then follows that

$$u_{t_i} = u_{t_{(i-1)}} + v_i - w_i. \tag{41}$$

So,

$$\begin{aligned} E[u_{t_3}^2] &= E[(u_{t_2} + v_3 - w_3)^2] && (42)\\ &= E[u_{t_2}^2 + v_3^2 + w_3^2 - 2w_3u_{t_2} + 2v_3u_{t_2} - 2v_3w_3] && (43)\\ &= E[u_{t_2}^2 + v_3^2 + w_3^2 - 2w_3u_{t_2} + 2v_3(u_{t_2} - w_3)]. && (44) \end{aligned}$$

But by Assumptions (B) and (D), v_3 is $-k\ sign(u_{t_2} - w_3)$. Furthermore, $E(w_i^2)$ is $1/\bar{\lambda}$. Further, as W is a martingale, $E[w_3|u_{t_2}] = 0$, and so $E[w_3 u_{t_2}] = 0$. So, (44) is

$$E[u_{t_2}^2] + k^2 + 1/\bar{\lambda} - 2kE[\mid u_{t_2} - w_3 \mid]. \tag{45}$$

Moreover, $(u_{t_2} - w_3)$ is u_{t_3-}, the right limit of u before the jump at t_3. As u is stationary, we may equate $E[u_{t_2}^2]$ and $E[u_{t_3}^2]$ to obtain the equality

$$E[\mid u_{t_3-} \mid] = \frac{k}{2}\left[\frac{1}{k^2\bar{\lambda}} + 1\right]. \tag{46}$$

But, conditional on $[X]_T$, $E\ [Y]_T = k^2\bar{\lambda}[X]_T$. As one could equally have looked at any two successive jumps (not only the second and third), the proposition follows.

B Proof of Proposition 3.3

The proposition states: Suppose that Assumptions (B), (C) and (D) of Theorem 2.1, as well as the Identification Assumption, hold. Uncorrelated Alternation then implies that Z has Ideal Error.

Proof First suppose that Y has Ideal Error. Then $Z = Y$ has Ideal Error trivially. Now, and for the rest of the proof, assume that Y does not have Ideal Error. If Q has first lag autocorrelation of zero then it is easily checked that

$$E(G_{t_{i+1}}|G_{t_i}) = E(G_{t_{i+1}}|G_{t_i}, G_{t_{(i-1)}}). \tag{47}$$

Therefore, by the Identification Assumption,

$$E(\epsilon_{t_{i+1}-}|G_{t_i}) = E(\epsilon_{t_{i+1}-}|G_{t_i}, G_{t_{(i-1)}}). \tag{48}$$

But then, as no jumps occurred between t_i and t_{i-1},

$$E(\epsilon_{t_i}|G_{t_i}) = E(\epsilon_{t_i}|G_{t_i}, G_{t_{(i-1)}}). \tag{49}$$

The Proposition now follows from Corollary B.1.

Corollary B.1 *Assume Assumptions (B), (C) and (D) of Theorem 2.1, and that Y does not have Ideal Error. Then Z has Ideal Error iff at each jump, timed t_i, $i > 1$,*

$$E(\epsilon_{t_i}|G_{t_i}) = E(\epsilon_{t_i}|G_{t_i}, G_{t_{(i-1)}}). \tag{50}$$

Proof. If Z has Ideal Error, then by Lemma B.2, for all t,

$$E\left(Z_t - X_t\right| \text{ the last two jumps in Y went up, then down }) = 0. \quad (51)$$

So, conditional on the two jumps in Y before t going up, then down

$$E\left(Y_t - X_t\right) = Y_t - Z_t. \quad (52)$$

So,

$$E\left(\epsilon_t\right| \text{ last 2 jumps in Y went up, then down}) = E\left(\epsilon_t\right| \text{ last jump in Y went down}). \quad (53)$$

The proposition now follows by the up-down symmetry of the model, considering exhaustively the four cases where prior to t:

the last 2 jumps in Y went up, then down,

the last 2 jumps in Y went up, then up,

the last 2 jumps in Y went down, then up, and

the last 2 jumps in Y went down, then down. ■

So Z has Ideal Error whenever conditioning not only on the last jump, but also on the last-but-one jump does not improve the best ergodic estimate of X_t given Y_t.

Lemma B.2 *Assume Assumptions (B), (C) and (D) of Theorem 2.1. Then for any t,*

$$E[Z]_T - [X]_T = 2(R-1)E[Y]_T p_A E\left(\frac{Z_t - X_t}{k}\right| \textit{ the last two jumps in Y went up, then down } \Bigg), \quad (54)$$

where p_A is the probability that a jump is an alternation.

Proof. See Appendix F. ■

C Proof of Proposition 3.4

The Proposition states: Suppose that Assumptions (B),(C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then,

$$E[A_T R - C_T] = 0, \quad (55)$$

and R has the Method of Moments estimator

$$\hat{R} = \frac{C_T}{A_T}. \quad (56)$$

(Define $\hat{R} = 0$ if $C_T = A_T = 0$).

The proof studies two cases in turn. The second case, where Y does not have Ideal Error, contains an important argument.

Case where Y has Ideal Error Then $R = 1$. By Proposition 3.1, the expected absolute value of ϵ_t just before a jump is k. Therefore, the expected value of ϵ_t conditional on Y just having jumped upwards is 0. The Identification Assumption implies that Y has equal probability of jumping up as down after this upwards jump. Given buy-sell symmetry, and as Q is uncorrelated, the ergodic probability that any given jump is an alternation is 0.5. Hence

$$E[\ A_T - C_T\ ] = 0. \tag{57}$$

Case where Y does not have Ideal Error Condition on $[X]_T$. Z has Ideal Error if

$$\begin{aligned} [X]_T &= E([Z]_T) & (58)\\ &= E(k^2(C_T + A_T R^2)). & (59) \end{aligned}$$

Also, R is defined by

$$\begin{aligned} [X]_T &= RE([Y]_T) & (60)\\ &= E(k^2 R(C_T + A_T)). & (61) \end{aligned}$$

Subtracting and dividing by k^2, we therefore have the moment condition,

$$E[\ \ (C_T + A_T R^2) - R(C_T + A_T)\ \ ] = 0. \tag{62}$$

Or, factorizing,

$$(R-1)\ E[\ \ (A_T R - C_T)\ \ ] = 0. \tag{63}$$

Since Y does not have Ideal Error, $R \neq 1$. Divide through by $(R-1)$:

$$E[\ A_T R - C_T\ ] = 0. \tag{64}$$

D Proof of Proposition 3.5

The proposition states : Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Condition on $[X]_T$, letting T be random. The following limit theory applies:

$$\lim_{\alpha\to 0}\sqrt{N_T}\left(\frac{\hat{R}[Y]_T}{[X]_T}-1\right)\sim N(0,UMU'), \tag{65}$$

where U is the pair $(1,\frac{(1+R)^2}{R})$ and V is the long-run variance matrix of the stationary time series of pairs, Π:

$$\Pi=\left\{\left(\frac{[X]_{t_i}-[X]_{t_{(i-1)}}}{E([X]_{t_i}-[X]_{t_{(i-1)}})}\ ,\ \frac{1+Q_i}{2}\right)\ :i\in\mathbb{N}\right\}. \tag{66}$$

Left hand term is the de-averaged elapsed QV in X between the $(i-1)$th and ith jumps in Y.

Proof Condition on $[X]_T$, and call it S, a business time. Let $\{s_1,s_2,s_3...\}$ be the business times of the observed jumps in Y, i.e. the times of the jumps in V. Then Π reduces to

$$\Pi=\left\{\left(\bar{\lambda}(s_i-s_{(i-1)}),\ \frac{1+Q_i}{2}\right)\ :i\in\mathbb{N}\right\}. \tag{67}$$

Note that the right-hand fraction takes the value 1 when Y alternates, and 0 when it continues. We are interested in the limit as $\alpha\to 0$ of

$$P(V^{\frac{1}{\alpha}}|W^\alpha)P(W). \tag{68}$$

First, however, consider the model,

$$P(V|W^\alpha)P(W). \tag{69}$$

This simply implies that for given α, V is observed until time S/α^2. In an abuse of notation, let N be the number of jumps before this time, of which let A be the number of alternations, and let C be the number of continuations. As $\alpha\to 0$, the number of jumps in V before time S/α^2 increases without bound, i.e. $N\to\infty$ with probability 1. By a standard central limit theorem, as S/α^2 is the sum of the durations between the observed jumps (in business time), ignoring the time after the last jump in the sample,

$$\lim_{\alpha\to 0}\sqrt{N}\left(\begin{pmatrix}\frac{S}{\alpha^2}\frac{\bar{\lambda}}{N}\\ \frac{A}{N}\end{pmatrix}-\begin{pmatrix}1\\ p_A\end{pmatrix}\right)\sim N(0,M), \tag{70}$$

where p_A is the probability that a jump is an alternation. Let

$$f:(x,y)\to(1-y)/xy. \tag{71}$$

Then f is differentiable in the positive quadrant and so by the Delta Method,

$$\lim_{\alpha \to 0} \sqrt{N} \left(\frac{NC\alpha^2}{S\bar{\lambda}A} - \frac{(1-p_A)}{p_A} \right) \sim N(0, df'Mdf), \tag{72}$$

where df is evaluated at $(1, p_A)'$. But the moment constraint in Proposition 3.4 implies that

$$\frac{(1-p_A)}{p_A} = R, \tag{73}$$

so, by simple calculation, the evaluated df is $-RU'$ and

$$\lim_{\alpha \to 0} \sqrt{N} \left(\frac{NC\alpha^2}{S\bar{\lambda}A} - R \right) \sim N(0, RUMU'R). \tag{74}$$

So,

$$\lim_{\alpha \to 0} \sqrt{N} \left(\frac{N(\alpha k)^2 C}{Sk^2\bar{\lambda}RA} - 1 \right) \sim N(0, UMU'). \tag{75}$$

Recall the following identities:

$$N(\alpha k)^2 = [V^{\frac{1}{\alpha}}]_S \ \ ; \ \ k^2\bar{\lambda}R = 1 \ \ ; \ \ S = [X]_T. \tag{76}$$

So, substituting these into (75),

$$\lim_{\alpha \to 0} \sqrt{N} \left(\frac{[V^{\frac{1}{\alpha}}]_S C}{[X]_T A} - 1 \right) \sim N(0, UMU'). \tag{77}$$

Return to the asymptotic theory of central interest, which takes the limit as $\alpha \to 0$ of

$$P(V^{\frac{1}{\alpha}} | W^{\alpha}) P(W). \tag{78}$$

Under this limit theory, $Y = V^{\frac{1}{\alpha}}_{[X]}$. Therefore $[V^{\frac{1}{\alpha}}]_S = [Y]_T$, $N = N_T$, $C = C_T$ and $A = A_T$. The proposition follows.

E Proof of Proposition 5.1

The proposition states : Suppose that Y evolves according to Sluggish Rounding. Then it has Uncorrelated Alternation. Furthermore, Π is an i.i.d. sequence and

$$UMU' = \frac{2}{3R}(1 + 4R + 2R^2). \tag{79}$$

Therefore UMU' may be estimated consistently by replacing R in (79) with $\hat{R} = C_T/A_T$.

Proof Condition on $[X]_T$. In business time, the duration between jumps is the time taken for a standard Brownian motion to exit the interval $(-k, Rk)$. A moment's thought reveals that Q and Π are then i.i.d. The probability that a Brownian motion starting at zero reaches the level Rk before the level $-k$ is known to be

$$\frac{k}{k+Rk}, \tag{80}$$

or $1/(1+R)$. The expected time to the first hit is Rk^2. This therefore also describes the probability that at a jump Z moves up (down) by Rk rather than down (up) by k, i.e. the probability that Y alternates, rather than continuing, and the expected duration in business time before this jump. The following formulae are derived from results recorded in Borodin and Salminen (1996):

The variance of the time between jumps is

$$\frac{1}{3}Rk^4(1+R^2) \tag{81}$$

So the variance of the normalized time between jumps, $\bar{\lambda}(s_i - s_{(i-1)})$, is

$$\frac{1+R^2}{3R} \tag{82}$$

The expected time between jumps, conditional on them alternating, is

$$\frac{2Rk^2(2R+1)}{3(R+1)} \tag{83}$$

Therefore the covariance of Q_i and $\bar{\lambda}(s_i - s_{(i-1)})$ is

$$-\frac{1-R}{3(1+R)} \tag{84}$$

The variance of Q_i is

$$\frac{R}{(1+R)^2} \tag{85}$$

These give the components of the matrix M. UMU' is now easily calculated.

F Proof of Proposition B.2

The proposition states : Assume Assumptions (B), (C) and (D) of Theorem 2.1. Condition on $[X]_T$. For any t,

$$E[Z]_T - [X]_T = 2(R-1)E[Y]_T p_A E\left(\frac{Z_t - X_t}{k} | \text{ the last two jumps in Y went up, then down}\right), \tag{86}$$

where p_A is the probability that a jump is an alternation.

Proof Let $S = [X]_T$ be known. Let $\tilde{Z}$ be $Z_{[X]^{-1}}$, i.e. $\tilde{Z}$ is Z as it evolves in business time. Define

$$\eta_s = \tilde{Z}_s - W_s. \tag{87}$$

So, η is the error in Z, as it evolves in business time. As $V - W$ is stationary, η is too. It follows the differential equation

$$d\eta_s = H_s N'_s - dW_s, \tag{88}$$

so that N' is the driving counting process of V. Say that the intensity of this counting process is λ. H is a process which takes value $\pm k$, and $\pm Rk$, depending on whether V is alternating or continuing, up or down. Then,

$$E((\eta_s + d\eta_s)^2) = E(\eta_s^2). \tag{89}$$

Therefore,

$$E(\eta_s^2 + 2\eta_s d\eta_s + d\eta_s^2) = E(\eta_s^2). \tag{90}$$

And

$$-2E(\eta_s d\eta_s) = E(d\eta_s^2). \tag{91}$$

Or,

$$-2E(\eta_s(H_s dN'_s - dW_s)) = E((H_s dN'_s - dW_s)^2). \tag{92}$$

So

$$-2E(\eta_s H_s \lambda_s)dt = E(H_s^2 \lambda_s)dt + dt. \tag{93}$$

Multiplying by $-S/ds$ and adding on a constant,

$$2SE(\eta_s H_s \lambda_s) + 2SE(H_s^2 \lambda_s) = SE(H_s^2 \lambda_s) - S. \tag{94}$$

Or,

$$2SE((\eta_s + H_s)H_s \lambda_s) = SE(H_s^2 \lambda_s) - S. \tag{95}$$

But the left hand side of this is

$$2S\bar{\lambda}E(\eta_s H_s | \text{ jump at t }) \tag{96}$$

While the right hand side is

$$E[Z]_T - [X]_T. \tag{97}$$

Putting this together, given the up-down symmetry of η,

$$E[Z]_T - [X]_T = 2S\bar{\lambda}E(\eta_s H_s | \ Y \text{ jumped up at t }). \tag{98}$$

But, $E[Y]_T = S\bar{\lambda}k^2$, so

$$E[Z]_T - [X]_T = \frac{2}{k^2}E[Y]_T E(\eta_s H_s | \ Y \text{ jumped up at t }). \tag{99}$$

So, writing p_A for the ergodic probability of alternation; and distinguishing the case of an alternation from that of a continuation in order to extract the magnitude of H from (99) we obtain

$$\begin{aligned}
& \frac{2}{k^2}E[Y]_T \left\{p_A RkE(\eta_s | \ Y \text{ alternated up at s }) + (1-p_A)kE(\eta_s | \ Y \text{ continued up at s })\right\} \\
= \ & \frac{2}{k}E[Y]_T E(\eta_s | \ Y \text{ jumped up at s }) + \frac{2}{k}E[Y]_T p_A(R-1)E(\eta_s | \ Y \text{ alternated up at s }) \\
= \ & 0 + \frac{2}{k}E[Y]_T p_A(R-1)E(\eta_s | \ Y \text{ alternated up at s }).
\end{aligned}$$

Therefore,

$$E[Z]_T - [X]_T = -\frac{2}{k}E[Y]_T(1-R)p_A E(\eta_s | \ Y \text{ alternated up at s }). \tag{100}$$

Or,

$$E[Z]_T - [X]_T = \frac{2}{k}E[Y]_T(1-R)p_A E(\eta_s | \ Y \text{ alternated down at s }). \tag{101}$$

G Proof of Corollary 5.2

The corollary states: suppose that the models $(X_1 + X_2, Y_1 + Y_2)$ and $(X_1 - X_2, Y_1 - Y_2)$ each satisfy the assumptions of Proposition 3.4 and, conditional on (X_1, X_2), Y_1 and Y_2 are independent. Write C^+ (A^+) for the number of continuations (alternations) in $Y_1 + Y_2$. Define C^- and A^- analogously for $Y_1 - Y_2$. Then

$$\frac{1}{4}\left(\frac{C^+}{A^+} - \frac{C^-}{A^-}\right)(\ [Y_1]_T + [Y_2]_T\ ) \tag{102}$$

is a consistent estimator of $[X_1, X_2]_T$.

Proof. Begin with the identity

$$[X_1, X_2]_T = \frac{1}{4}\left([X_1 + X_2]_T - [X_1 - X_2]_T\right). \tag{103}$$

Note that as Y_1 and Y_2 are conditionally independent pure jump processes,

$$[Y_1]_T + [Y_2]_T = [Y_1 + Y_2]_T = [Y_1 - Y_2]_T. \tag{104}$$

As $(X_1 + X_2, Y_1 + Y_2)$ and $(X_1 - X_2, Y_1 - Y_2)$ satisfy the assumptions of Proposition 3.4,

$$[Y_1 + Y_2]_T \frac{C^+}{A^+} \text{ and } [Y_1 - Y_2]_T \frac{C^-}{A^-} \tag{105}$$

are consistent estimators of

$$[X_1 + X_2]_T \text{ and } [X_1 - X_2]_T. \tag{106}$$

The corollary now follows by a straightforward substitution. ■